Wells and Hart

SOLID GEOMETRY

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H.-F. SOLID GEOMETRY.
W. P. I

PREFACE

In addition to the features of the Plane Geometry, which are emphasized in the Solid as well, the chief characteristic of this book is the establishment, at every point, of *the vital relation between the Solid and the Plane Geometry.* Many theorems in Solid Geometry have been proved, and many problems have been solved, by reducing them to a plane, and simply applying the corresponding principle of Plane Geometry. Again, many proofs of Plane Geometry have been made to serve as proofs of corresponding theorems in Solid Geometry by merely making the proper changes in terms used. (See §§ 703, 786, 794, 813, 853, 924, 951, 955, 961, etc.)

Other special features of the book may be summarized as follows:

The student is given every possible aid in forming his early space concepts. In the early work in Solid Geometry, the average student experiences difficulty in fully comprehending space relations; that is, in *seeing* geometric figures in space. The student is aided in overcoming this difficulty by the introduction of many easy and practical questions and exercises, as well as by being encouraged to *make* his figures. (See § 605.) As a further aid in this direction, reproductions of models made by students themselves are shown in a group (p. 302), and at various points throughout Book VI.

The student's fund of knowledge is constantly drawn upon. In the many questions, suggestions, and exercises, his knowledge of the things about him has been constantly appealed to. Especially is this true of the work on the sphere, where the

student's knowledge of mathematical geography has been appealed to in making clear the terms and the relations of figures connected with the sphere.

The treatment of the Solid Geometry is logical. The same logical rigor that characterizes the demonstrations in the Plane Geometry is used consistently throughout the Solid. If a postulate is needed to make a proof complete, it is clearly stated, as in § 615. In the mensuration of the prism and the pyramid, the same general plan has been followed as that used in Book IV; in the mensuration of the cylinder, the cone, and the sphere, the method pursued is similar to that used in the mensuration of the circle.

More proofs and parts of proofs are left to the student in the Solid, than in the Plane Geometry; but in every case in which the proof is not complete, the incompleteness is specifically stated.

The treatment of the polyhedral angle (p. 336), *of the prism* (p. 345), *and of the pyramid* (p. 350), *is similar to that of the cylinder and the cone.* This is in accordance with the recommendations of the leading Mathematical Associations throughout the country.

The complete collection of formulas of Solid Geometry at the end of the book, it is hoped, will be found helpful to teacher and student alike.

The grateful acknowledgment of the authors is due to many friends for helpful suggestions; especially to Miss Grace A. Bruce, of the Wadleigh High School, New York; to Mr. Edward B. Parsons, of the Boys' High School, Brooklyn; and to Professor McMahon, of Cornell University.

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SYMBOLS AND ABBREVIATIONS

$=$	equals, equal to, is equal to.
$\neq$	does not equal.
$>$	greater than, is greater than.
$<$	less than, is less than.
$\backsimeq$	equivalent, equivalent to, is equivalent to.
$\sim$	similar, similar to, is similar to.
$\stackrel{\propto}{=}$	is measured by.
$\perp$	perpendicular, perpendicular to, is perpendicular to.
$\perp$s	perpendiculars.
$\parallel$	parallel, parallel to, is parallel to.
$\parallel$s	parallels.
. . .	and so on (sign of continuation).
$\because$	since.
$\therefore$	therefore.
$\frown$	arc; $\overset{\frown}{AB}$, arc AB.
▱, ▱s	parallelogram, parallelograms.
$\odot$, Ⓢ	circle, circles.
$\angle$, $\angle$s	angle, angles.
$\triangle$, $\triangle$s	triangle, triangles.

rt.	right.
str.	straight.
ext.	exterior.
int.	interior.
alt.	alternate.
def.	definition.
ax.	axiom.
post.	postulate.
hyp.	hypothesis.
prop.	proposition.
prob.	problem.
th.	theorem.
cor.	corollary.
cons.	construction.
ex.	exercise.
fig.	figure.
iden.	identity.
comp.	complementary.
sup.	supplementary.
adj.	adjacent.
homol.	homologous.

Q.E.D. Quod erat demonstrandum, *which was to be proved.*

Q.E.F. Quod erat faciendum, *which was to be done.*

The signs $+$, $-$, $\times$, $\div$ have the same meanings as in algebra.

REFERENCES TO THE PLANE GEOMETRY

Note. The following definitions, theorems, etc., from the Plane Geometry which are referred to in the Solid Geometry are here collected for the convenience of the student.

(The numbers below refer to articles in the Plane Geometry.)

18. Def. Two geometric figures are **equal** if they can be made to coincide.

26. Two intersecting straight lines can have only one point in common; *i.e.* two intersecting straight lines **determine** a point.

34. Def. A **plane surface** (or plane) is a surface of unlimited extent such that whatever two of its points are taken, a straight line joining them will lie wholly in the surface.

ASSUMPTIONS

54. 1. Things equal to the same thing, or to equal things, are equal to each other.

2. If equals are added to equals, the sums are equal.

3. If equals are subtracted from equals, the remainders are equal.

4. If equals are added to unequals, the sums are unequal in the same order.

5. If equals are subtracted from unequals, the remainders are unequal in the same order.

6. If unequals are subtracted from equals, the remainders are unequal in the reverse order.

7. (*a*) If equals are multiplied by equals, the products are equal; (*b*) if unequals are multiplied by equals, the products are unequal in the same order.

8. (*a*) If equals are divided by equals, the quotients are equal; (*b*) if unequals are divided by equals, the quotients are unequal in the same order.

9. If unequals are added to unequals, the less to the less and the greater to the greater, the sums are unequal in the same order.

10. If three magnitudes of the same kind are so related that the first is greater than the second, and the second greater than the third, then the first is greater than the third.

11. The whole is equal to the sum of all its parts.

12. The whole is greater than any of its parts.

13. Like powers of equal numbers are equal, and like roots of equal numbers are equal.

14. **Transference postulate.** Any geometric figure may be moved from one position to another without change of size or shape.

15. **Straight line postulate I.** A straight line may be drawn from any one point to any other.

16. **Straight line postulate II.** A line segment may be prolonged indefinitely at either end.

17. **Revolution postulate.** A straight line may revolve in a plane, about a point as a pivot, and when it does revolve continuously from one position to another, it passes once and only once through every intermediate position.

62. At every point in a straight line there exists only one perpendicular to the line.

63. At every point in a straight line there exists one and only one perpendicular to the line.

65. If one straight line meets another straight line, the sum of the two adjacent angles is two right angles.

76. If two adjacent angles are supplementary, their exterior sides are collinear.

77. If two straight lines intersect, the vertical angles are equal.

92. Def. A polygon of three sides is called a **triangle**; one of four sides, a **quadrilateral**; one of five sides, a **pentagon**; one of six sides, a **hexagon**; and so on.

105. Two triangles are equal if a side and the two adjacent angles of one are equal respectively to a side and the two adjacent angles of the other.

107. Two triangles are equal if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.

110. Homologous parts of equal figures are equal.

111. The base angles of an isosceles triangle are equal.

116. Two triangles are equal if the three sides of one are equal respectively to the three sides of the other.

122. Circle postulate. A circle may be constructed having any point as center, and having a radius equal to any finite line.

124. To construct an equilateral triangle, with a given line as side.

134. Every point in the perpendicular bisector of a line is equidistant from the ends of that line.

139. Every point equidistant from the ends of a line lies in the perpendicular bisector of that line.

142. Two points each equidistant from the ends of a line determine the perpendicular bisector of the line.

148. To construct a perpendicular to a given straight line at a given point in the line.

149. From a point outside a line to construct a perpendicular to the line.

153. If one side of a triangle is prolonged, the exterior angle formed is greater than either of the remote interior angles.

154. From a point outside a line there exists only one perpendicular to the line.

156. If two sides of a triangle are unequal, the angle opposite the greater side is greater than the angle opposite the less side.

161. (*a*) In the use of the indirect method the student should give, as argument 1, all the suppositions of which the case he is considering admits, including the conclusion. As reason 1 the number of such possible suppositions should be cited.

(*b*) As a reason for the last step in the argument he should state which of these suppositions have been proved false.

167. The sum of any two sides of a triangle is greater than the third side.

168. Any side of a triangle is less than the sum and greater than the difference of the other two.

173. If two triangles have two sides of one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.

178. Parallel line postulate. Two intersecting straight lines cannot both be parallel to the same straight line.

179. The following form of this postulate is sometimes more convenient to quote: Through a given point there exists only one line parallel to a given line.

180. If two straight lines are parallel to a third straight line, they are parallel to each other.

187. If two straight lines are perpendicular to a third straight line, they are parallel to each other.

190. If two parallel lines are cut by a transversal, the corresponding angles are equal.

192. If two parallel lines are cut by a transversal, the sum of the two interior angles on the same side of the transversal is two right angles.

193. A straight line perpendicular to one of two parallels is perpendicular to the other also.

194. If two straight lines are cut by a transversal making the sum of the two interior angles on the same side of the transversal not equal to two right angles, the lines are not parallel.

198. Two angles whose sides are parallel, each to each, are either equal or supplementary.

206. In a triangle there can be but one right angle or one obtuse angle.

209. Two right triangles are equal if the hypotenuse and an acute angle of one are equal respectively to the hypotenuse and an acute angle of the other.

211. Two right triangles are equal if the hypotenuse and a side of one are equal respectively to the hypotenuse and a side of the other.

215. An exterior angle of a triangle is equal to the sum of the two remote interior angles.

216. The sum of all the angles of any polygon is twice as many right angles as the polygon has sides, less four right angles.

220. Def. A **parallelogram** is a quadrilateral whose opposite sides are parallel.

228. Def. Any side of a parallelogram may be regarded as its **base**, and the line drawn perpendicular to the base from any point in the opposite side is then the **altitude.**

232. The opposite sides of a parallelogram are equal.

234. Parallel lines intercepted between the same parallel lines are equal.

240. If two opposite sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

252. The two perpendiculars to the sides of an angle from any point in its bisector are equal.

253. Every point in the bisector of an angle is equidistant from the sides of the angle.

258. The bisectors of the angles of a triangle are concurrent in a point which is equidistant from the three sides of the triangle.

276. **Def.** A **circle** is a plane closed figure whose boundary is a curve such that all straight lines to it from a fixed point within are equal.

279. (*a*) All radii of the same circle are equal.
(*b*) All radii of equal circles are equal.
(*c*) All circles having equal radii are equal.

297. Four right angles contain 360 angle degrees, and four right angles at the center of a circle intercept a complete circumference; therefore, a circumference contains 360 arc degrees. Hence, a semicircumference contains 180 arc degrees.

298. In equal circles, or in the same circle, if two chords are equal, they subtend equal arcs; conversely, if two arcs are equal, the chords that subtend them are equal.

307. In equal circles, or in the same circle, if two chords are equal, they are equally distant from the center; conversely, if two chords are equally distant from the center, they are equal.

308. In equal circles, or in the same circle, if two chords are unequal, the greater chord is at the less distance from the center.

309. **Note.** The student should always give the full statement of the substitution made; for example, "Substituting AE for its equal CD."

310. In equal circles, or in the same circle, if two chords are unequally distant from the center, the chord at the less distance is the greater.

313. A tangent to a circle is perpendicular to the radius drawn to the point of tangency.

314. A straight line perpendicular to a radius at its outer extremity is tangent to the circle.

321. To inscribe a circle in a given triangle.

323. To circumscribe a circle about a given triangle.

324. Three points not in the same straight line determine a circle.

328. If two circumferences intersect, their line of centers bisects their common chord at right angles.

335. Def. To **measure** a quantity is to find how many times it contains another quantity of the same kind. The result of the measurement is a **number** and is called the **numerical measure**, or **measure-number**, of the quantity which is measured. The measure employed is called the **unit of measure.**

337. Def. Two quantities are **commensurable** if there exists a measure that is contained an integral number of times in each. Such a measure is called a **common measure** of the two quantities.

339. Def. Two quantities are **incommensurable** if there exists no measure that is contained an integral number of times in each.

341. Def. The **ratio of two geometric magnitudes** may be defined as the quotient of their measure-numbers, when the same measure is applied to each.

349. Def. If a variable approaches a constant in such a way that the difference between the variable and the constant may be made to become and remain smaller than any fixed number previously assigned, however small, the constant is called the **limit** of the variable.

355. If two variables are always equal, and if each approaches a limit, then their limits are equal.

358. An angle at the center of a circle is measured by its intercepted arc.

362. (*a*) In equal circles, or in the same circle, equal angles are measured by equal arcs; conversely, equal arcs measure equal angles.

(*b*) The measure of the $\left\{\begin{matrix}\text{sum}\\ \text{difference}\end{matrix}\right\}$ of two angles is equal to the $\left\{\begin{matrix}\text{sum}\\ \text{difference}\end{matrix}\right\}$ of the measures of the angles.

(*c*) The measure of any multiple of an angle is equal to that same multiple of the measure of the angle.

373. To construct a tangent to a circle from a point outside.

399. If four numbers are in proportion, they are in proportion by **division**; that is, the difference of the first two terms is to the first (or second) term as the difference of the last two terms is to the third (or fourth) term.

401. In a series of equal ratios the sum of any number of antecedents is to the sum of the corresponding consequents as any antecedent is to its consequent.

409. A straight line parallel to one side of a triangle divides the other two sides proportionally.

410. A straight line parallel to one side of a triangle divides the other two sides into segments which are proportional.

419. Def. Two **polygons** are **similar** if they are mutually equiangular and if their sides are proportional.

420. Two triangles which are mutually equiangular are similar.

422. Two right triangles are similar if an acute angle of one is equal to an acute angle of the other.

424. (1) Homologous angles of similar triangles are equal.

(2) Homologous sides of similar triangles are proportional.

(3) Homologous sides of similar triangles are the sides opposite equal angles.

435. In two similar triangles any two homologous altitudes have the same ratio as any two homologous sides.

438. If two polygons are composed of the same number of triangles, similar each to each and similarly placed, the polygons are similar.

443. In a right triangle, if the altitude upon the hypotenuse is drawn:

I. The square of the altitude is equal to the product of the segments of the hypotenuse.

II. The square of either side is equal to the product of the whole hypotenuse and the segment of the hypotenuse adjacent to that side.

444. If from any point in the circumference of a circle a perpendicular to a diameter is drawn, and if chords are drawn from the point to the ends of the diameter:

I. The perpendicular is a mean proportional between the segments of the diameter.

II. Either chord is a mean proportional between the whole diameter and the segment of the diameter adjacent to the chord.

478. The area of a square is equal to the square of its side.

479. Any two rectangles are to each other as the products of their bases and their altitudes.

480. (*a*) Two rectangles having equal bases are to each other as their altitudes, and (*b*) two rectangles having equal altitudes are to each other as their bases.

481. The area of a parallelogram equals the product of its base and its altitude.

482. Parallelograms having equal bases and equal altitudes are equivalent.

483. Any two parallelograms are to each other as the products of their bases and their altitudes.

484. (*a*) Two parallelograms having equal bases are to each other as their altitudes, and (*b*) two parallelograms having equal altitudes are to each other as their bases.

485. The area of a triangle equals one half the product of its base and its altitude.

491. The area of a triangle is equal to one half the product of its perimeter and the radius of the inscribed circle.

492. The area of any polygon circumscribed about a circle is equal to one half its perimeter multiplied by the radius of the inscribed circle.

498. Two triangles which have an angle of one equal to an angle of the other are to each other as the products of the sides including the equal angles.

503. Two similar triangles are to each other as the squares of any two homologous sides.

517. If the circumference of a circle is divided into any number of equal arcs: (*a*) the chords joining the points of division form a regular polygon inscribed in the circle; (*b*) tangents drawn at the points of division form a regular polygon circumscribed about the circle.

538. The perimeters of two regular polygons of the same number of sides are to each other as their radii or as their apothems.

541. I. The perimeter and area of a regular polygon inscribed in a circle are less, respectively, than the perimeter and area of the regular inscribed polygon of twice as many sides.

II. The perimeter and area of a regular polygon circumscribed about a circle are greater, respectively, than the perimeter and area of the regular circumscribed polygon of twice as many sides.

543. By repeatedly doubling the number of sides of a regular polygon inscribed in a circle, and making the polygons always regular:

I. The apothem can be made to differ from the radius by less than any assigned value.

II. The square of the apothem can be made to differ from the square of the radius by less than any assigned value.

546. By repeatedly doubling the number of sides of regular circumscribed and inscribed polygons of the same number of sides, and making the polygons always regular:

I. Their perimeters approach a common limit.

II. Their areas approach a common limit.

550. Def. The **length of a circumference** is the common limit which the successive perimeters of inscribed and circumscribed regular polygons (of 3, 4, 5, etc., sides) approach as the number of sides is successively increased and each side approaches zero as a limit.

556. Any two circumferences are to each other as their radii.

558. Def. The **area of a circle** is the common limit which the successive areas of inscribed and circumscribed regular polygons approach as the number of sides is successively increased and each side approaches zero as a limit.

559. The area of a circle is equal to one half the product of its circumference and its radius.

586. If a variable can be made less than any assigned value, the quotient of the variable by any constant, except zero, can be made less than any assigned value.

587. If a variable can be made less than any assigned value, the product of that variable and a decreasing value may be made less than any assigned value.

590. The limit of the product of a variable and a constant, not zero, is the limit of the variable multiplied by the constant.

592. If two variables approach finite limits, not zero, then the limit of their product is equal to the product of their limits.

593. If each of any finite number of variables approaches a finite limit, not zero, then the limit of their product is equal to the product of their limits.

594. If two related variables are such that one is always greater than the other, and if the greater continually decreases while the less continually increases, so that the difference between the two may be made as small as we please, then the two variables have a common limit which lies between them.

599. An angle can be bisected by only one line.

SOLID GEOMETRY

BOOK VI

LINES, PLANES, AND ANGLES IN SPACE

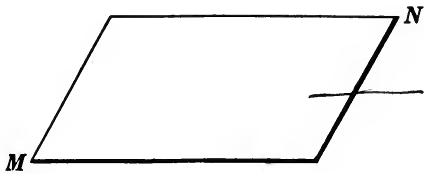
602. Def. Solid geometry or the **geometry of space** treats of figures whose parts are not all in the same plane. (For definition of *plane* or *plane surface*, see § 34.)

603. From the definition of a plane it follows that:

(*a*) *If two points of a straight line lie in a plane, the whole line lies in that plane.*

(*b*) *A straight line can intersect a plane in not more than one point.*

604. Since a plane is unlimited in its two dimensions (length and breadth) only a portion of it can be shown in a figure. This is usually represented by a quadrilateral drawn as a parallelogram. Thus



MN represents a plane. Sometimes, however, conditions make it necessary to represent a plane by a figure other than a parallelogram, as in § 617.

Ex. 1152. Draw a rectangle freehand which is supposed to lie: (*a*) in a vertical plane; (*b*) in a horizontal plane. May the four angles of the rectangle of (*a*) be drawn equal? those of the rectangle of (*b*)?

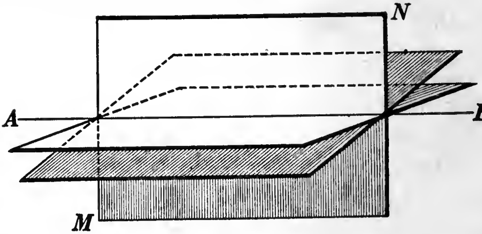
605. Note. In the figures in solid geometry dashed lines will be used to represent all auxiliary lines and lines that are not supposed to be visible but which, for purposes of proof, are represented in the figure. All other lines will be continuous. In the earlier work in solid geometry the student may experience difficulty in *imagining* the figures. If so, he may find it a great help, for a time at least, to *make* the figures. By using pasteboard to represent planes, thin sticks of wood or stiff wires to represent lines perpendicular to a plane, and strings to represent oblique lines, any figure may be actually made with a comparatively small expenditure of time and with practically no expense. For reproductions of models actually made by high school students, see group on p. 302; also §§ 622, 633, 678, 756, 762, 770, 797.

606. Assumption 20. Revolution postulate. *A plane may revolve about a line in it as an axis, and as it does so revolve, it can contain any particular point in space in one and only one position.*

607. From the revolution postulate it follows that:

Through a given straight line any number of planes may be passed.

For, as plane MN revolves about AB as an axis (§ 606) it may occupy an unlimited number of positions each of which will represent a different plane through AB.



608. Def. A **plane** is said to be **determined** by given conditions if that plane and no other plane fulfills those conditions.

609. From §§ 607 and 608 it is seen that:

A straight line does not determine a plane.

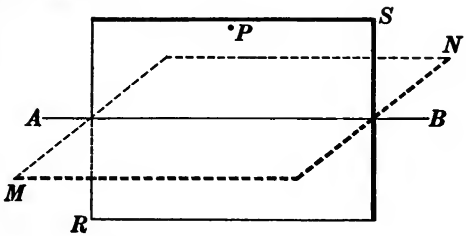
Ex. 1153. How many planes may be passed through any two points in space? why?

Ex. 1154. At a point P in a given straight line AB in space, construct a line perpendicular to AB. How many such lines can be drawn?

LINES AND PLANES

Proposition I. Theorem

610. *A plane is determined by a straight line and a point not in the line.*



Given line AB and P, a point not in AB.

To prove that AB and P determine a plane.

Argument	Reasons
1. Through AB pass any plane, as MN.	1. § 607.
2. Revolve plane MN about AB as an axis until it contains point P. Call the plane in this position RS.	2. § 606.
3. Then plane RS contains line AB and point P.	3. Arg. 2.
4. Furthermore, in no other position can plane MN, in its rotation about AB, contain point P.	4. § 606.
5. $\therefore$ RS is the only plane that can contain AB and P.	5. Arg. 4.
6. $\therefore$ AB and P determine a plane. Q.E.D.	6. § 608.

611. Cor. I. *A plane is determined by three points not in the same straight line.*

Hint. Let A, B, and C be the three given points. Join A and B by a straight line, and apply § 610.

612. Cor. II. *A plane is determined by two intersecting straight lines.*

613. Cor. III. *A plane is determined by two parallel straight lines.*

Ex. 1155. Given line AB in space, and P a point not in AB. Construct, through P, a line perpendicular to AB.

Ex. 1156. Hold two pencils so that a plane can be passed through them. In how many ways can this be done, assuming that the pencils are lines? why?

Ex. 1157. Can two pencils be held so that no plane can be passed through them? If so, how?

Ex. 1158. In measuring wheat with a half bushel measure, the measure is first heaped, then a straightedge is drawn across the top. Why is the measure then even full?

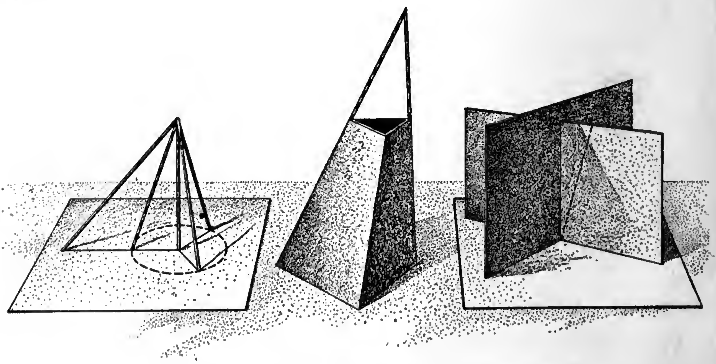
Ex. 1159. Why is a surveyor's transit or a photographer's camera always supported on three legs rather than on two or four?

Ex. 1160. How many planes are determined by four straight lines, no three of which lie in the same plane, if the four lines intersect: (1) at a common point? (2) at four different points?

---

614. Def. The **intersection of two surfaces** is the locus of all points common to the two surfaces.

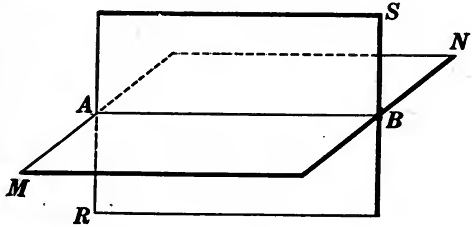
615. Assumption 21. Postulate. *Two planes having one point in common also have another point in common.*



Reproduced from Models made by High School Students

PROPOSITION II. THEOREM

616. *If two planes intersect, their intersection is a straight line.*



Given intersecting planes *MN* and *RS*.

To prove the intersection of *MN* and *RS* a str. line.

ARGUMENT	REASONS
1. Let *A* and *B* be any two points common to the two planes *MN* and *RS*.	1. § 615.
2. Draw str. line *AB*.	2. § 54, 15.
3. Since both *A* and *B* lie in plane *MN*, str. line *AB* lies in plane *MN*.	3. § 603, *a*.
4. Likewise str. line *AB* lies in plane *RS*.	4. § 603, *a*.
5. Furthermore no point outside of *AB* can lie in both planes.	5. § 610.
6. ∴ *AB* is the intersection of planes *MN* and *RS*.	6. § 614.
7. But *AB* is a str. line.	7. Arg. 2.
8. ∴ the intersection of *MN* and *RS* is a str. line. Q.E.D.	8. Args. 6 and 7.

Ex. 1161. Is it possible for more than two planes to intersect in a straight line? Explain.

Ex. 1162. By referring to §§ 26 and 608, give the meaning of the expression, "Two planes determine a straight line."

Ex. 1163. Is the statement in Ex. 1162 always true? Give reasons for your answer.

Proposition III. Theorem

617. *If three planes, not passing through the same line, intersect each other, their three lines of intersection are concurrent, or else they are parallel, each to each.*

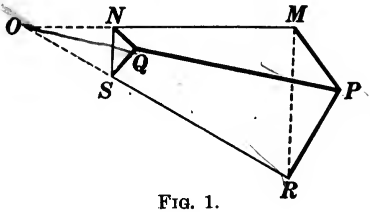


Fig. 1.

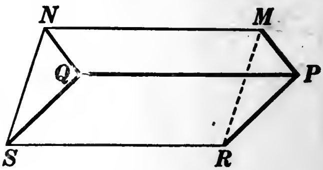


Fig. 2.

Given planes MQ, PS, and RN intersecting each other in lines MN, PQ, and RS; also:

I. **Given** MN and RS intersecting at O (Fig. 1).

To prove MN, PQ, and RS concurrent.

Argument	Reasons
1. $\because O$ is in line MN, it lies in plane MQ.	1. § 603, *a*.
2. $\because O$ is in line RS, it lies in plane PS.	2. § 603, *a*.
3. $\therefore O$, lying in planes MQ and PS, must lie in their intersection, PQ.	3. § 614.
4. $\therefore PQ$ passes through O; *i.e.* MN, PQ, and RS are concurrent in O. Q.E.D.	4. Arg. 3.

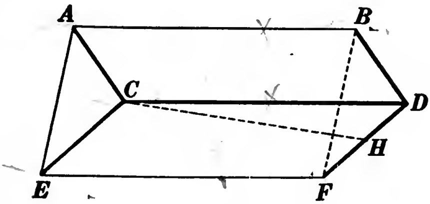
II. **Given** $MN \parallel RS$ (Fig. 2).

To prove $PQ \parallel MN$ and RS.

Argument	Reasons
1. PQ and MN are either $\parallel$ or not $\parallel$.	1. § 161, *a*.
2. Suppose that PQ intersects MN; then MN also intersects RS.	2. § 617, I.
3. But this is impossible, for $MN \parallel RS$.	3. By hyp.
4. $\therefore PQ \parallel MN$.	4. § 161, *b*.
5. Likewise $PQ \parallel RS$. Q.E.D.	5. By steps similar to 1–4.

a plane is determined 211

618. Cor. *If two straight lines are parallel to a third straight line, they are parallel to each other.*



Given lines AB and CD, each $\parallel EF$.

To prove $AB \parallel CD$.

Argument	Reasons
1. AB and CD are either $\parallel$ or not $\parallel$.	1. § 161, *a*.
2. Through AB and EF pass plane AF, and through CD and EF pass plane CF.	2. § 613.
3. Pass a third plane through AB and point C, as plane BC.	3. § 610.
4. Suppose that AB is not $\parallel CD$; then plane BC will intersect plane CF in some line other than CD, as CH.	4. § 613.
5. Then $CH \parallel EF$.	5. § 617, II.
6. But $CD \parallel EF$.	6. By hyp.
7. $\therefore CH$ and CD, two straight lines in plane CF, are both $\parallel EF$.	7. Args. 5 and 6.
8. This is impossible.	8. § 178.
9. $\therefore AB \parallel CD$. Q.E.D.	9. § 161, *b*.

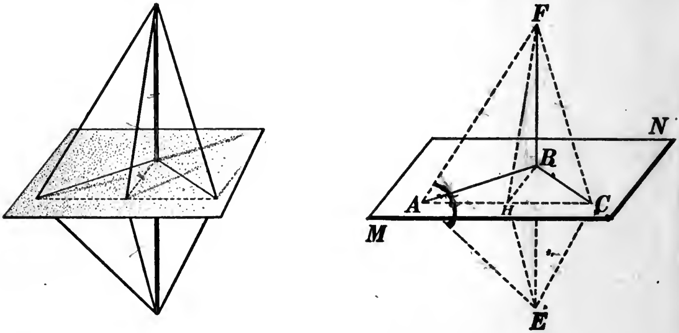
619. Def. A **straight line is perpendicular to a plane** if it is perpendicular to *every* straight line in the plane passing through the point of intersection of the given line and plane.

620. Def. A **plane is perpendicular to a straight line** if the line is perpendicular to the plane.

621. Def. If a line is perpendicular to a plane, its point of intersection with the plane is called the **foot of the perpendicular.**

Proposition IV. Theorem

622. *If a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, it is perpendicular to the plane of those lines.*



Given str. line $FB \perp AB$ and to BC at B, and plane MN containing AB and BC.

To prove $FB \perp$ plane MN.

Outline of Proof

1. In plane MN draw AC; through B draw any line, as BH, meeting AC at H.

2. Prolong FB to E so that $BE = FB$; draw AF, HF, CF, AE, HE, CE.

3. AB and BC are then $\perp$ bisectors of FE; *i.e.* $FA = AE$, $FC = CE$.

4. Prove $\triangle AFC = \triangle EAC$; then $\angle HAF = \angle EAH$.

5. Prove $\triangle HAF = \triangle EAH$; then $HF = HE$.

6. $\therefore BH \perp FE$; *i.e.* $FB \perp BH$, any line in plane MN passing through B.

7. $\therefore$ $FB \perp MN$.

623. Cor. *All the perpendiculars that can be drawn to a straight line at a given point in the line lie in a plane perpendicular to the line at the given point.*

Proposition V. Problem

624. *Through a given point to construct a plane perpendicular to a given line.*

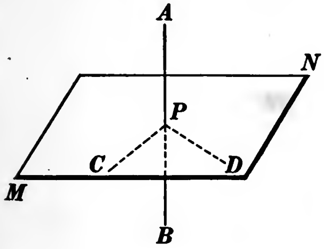


Fig. 1.

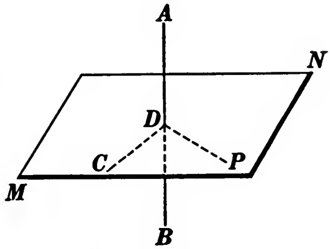


Fig. 2.

Given point *P* and line *AB*.

To construct, through *P*, a plane $\perp$ *AB*.

I. Construction

1. Through line *AB* and point *P* pass a plane, as *APD* (in Fig. 1, any plane through *AB*). §§ 607, 610.
2. In plane *APD* construct *PD*, through *P*, $\perp$ *AB*. §§ 148, 149.
3. Through *AB* pass a second plane, as *ABC*. § 607.
4. In plane *ABC*, through the foot of *PD*, construct a $\perp$ to *AB* (*PC* in Fig. 1, *DC* in Fig. 2). § 148.
5. Plane *MN*, determined by *C*, *D*, and *P*, is the plane required.

II. The proof is left as an exercise for the student.

Hint. Apply § 623.

III. The discussion will be given in § 625.

Ex. 1164. Tell how to test whether or not a flag pole is erect.

Ex. 1165. Lines *AB* and *CD* are each perpendicular to line *EF*. Are *AB* and *CD* necessarily parallel? Explain. Do they necessarily lie in the same plane? why or why not?

Proposition VI. Theorem

625. *Through a given point there exists only one plane perpendicular to a given line.*

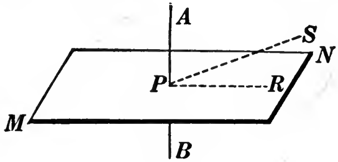


Fig. 1.

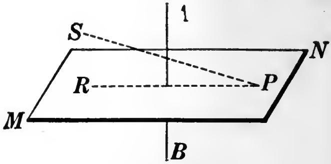


Fig. 2.

Given plane MN, through P, $\perp AB$.

To prove MN the only plane through $P \perp AB$.

Argument Only

1. Either MN is the only plane through $P \perp AB$ or it is not.
2. In MN draw a line through P intersecting line AB, as PR.
3. Let plane determined by AB and PR be denoted by APR.
4. Suppose that there exists another plane through $P \perp AB$; let this second plane intersect plane APR in line PS.
5. Then $AB \perp PR$ and also PS; *i.e.* PR and PS are $\perp AB$.
6. This is impossible.
7. $\therefore$ MN is the only plane through $P \perp AB$. Q.E.D.

---

626. Question. In Fig. 2, explain why $AB \perp PS$.

627. §§ 624 and 625 may be combined in one statement:

Through a given point there exists one and only one plane perpendicular to a given line.

628. Cor. I. *The locus of all points in space equidistant from the extremities of a straight line segment is the plane perpendicular to the segment at its mid-point.*

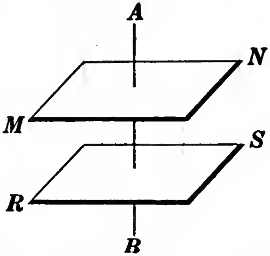
629. Def. A **straight line** is **parallel** to a plane if the straight line and the plane cannot meet.

630. Def. A **straight line** is **oblique** to a plane if it is neither perpendicular nor parallel to the plane.

631. Def. Two **planes** are **parallel** if they cannot meet.

PROPOSITION VII. THEOREM

632. *Two planes perpendicular to the same straight line are parallel.*



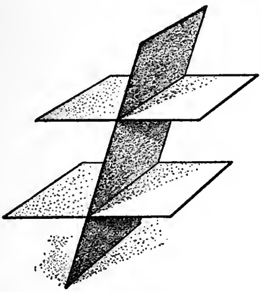
Given planes *MN* and *RS*, each ⊥ line *AB*.

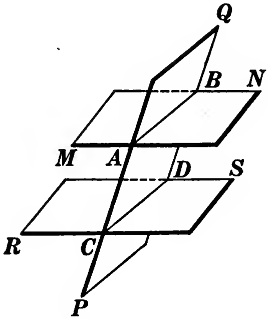
To prove *MN* ∥ *RS*.

HINT. Use indirect proof. Compare with § 187.

PROPOSITION VIII. THEOREM

633. *If a plane intersects two parallel planes, the lines of intersection are parallel.*





Given ∥ planes *MN* and *RS*, and any plane *PQ* intersecting *MN* and *RS* in *AB* and *CD*, respectively.

To prove *AB* ∥ *CD*.

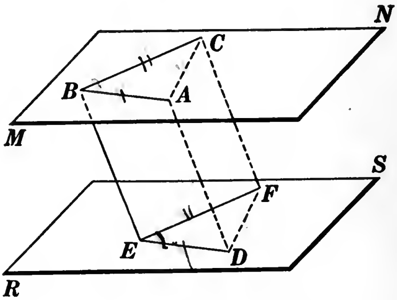
HINT. Show that *AB* and *CD* cannot meet.

634. Cor. I. *Parallel lines intercepted between the same parallel planes are equal.* (HINT. Compare with § 234.)

Ex. 1166. State the converse of Prop. VIII. Is it true?

Proposition IX. Theorem

635. *If two angles, not in the same plane, have their sides parallel respectively, and lying on the same side of the line joining their vertices, they are equal.**



Given $\angle ABC$ in plane MN and $\angle DEF$ in plane RS with BA and $BC \parallel$ respectively to ED and EF, and lying on the same side of line BE.

To prove $\angle ABC = \angle DEF$.

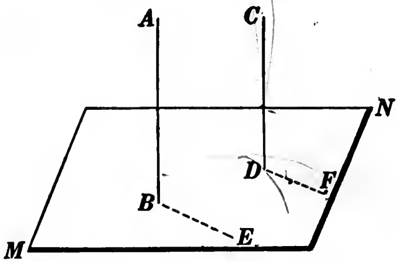
Argument	Reasons
1. Measure off $BA = ED$ and $BC = EF$.	1. § 122.
2. Draw AD, CF, AC, and DF.	2. § 54, 15.
3. $BA \parallel ED$ and $BC \parallel EF$.	3. By hyp.
4. Then $ADEB$ and $CFEB$ are ▱s.	4. § 240.
5. $\therefore AD = BE$ and $CF = BE$.	5. § 232.
6. $\therefore AD = CF$.	6. § 54, 1.
7. Also $AD \parallel BE$ and $CF \parallel BE$.	7. § 220.
8. $\therefore AD \parallel CF$.	8. § 618.
9. $\therefore ACFD$ is a ▱.	9. § 240.
10. $\therefore AC = DF$.	10. § 232.
11. But $BA = ED$ and $BC = EF$.	11. Arg. 1.
12. $\therefore \triangle ABC = \triangle DEF$.	12. § 116.
13. $\therefore \angle ABC = \angle DEF$. Q.E.D.	13. § 110.

Ex. 1167. Prove Prop. IX if the angles lie on opposite sides of BE.

* It will also be seen (§ 645) that the planes of these angles are parallel.

Proposition X. Theorem

636. *If one of two parallel lines is perpendicular to a plane, the other also is perpendicular to the plane.*

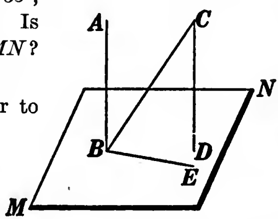


Given $AB \parallel CD$ and $AB \perp$ plane MN.

To prove $CD \perp$ plane MN.

Argument	Reasons
1. Through D draw any line in plane MN, as DF.	1. § 54, 15.
2. Through B draw BE in plane $MN \parallel DF$.	2. § 179.
3. Then $\angle ABE = \angle CDF$.	3. § 635.
4. But $\angle ABE$ is a rt. $\angle$.	4. § 619.
5. $\therefore \angle CDF$ is a rt. $\angle$; *i.e.* $CD \perp DF$, any line in plane MN through D.	5. § 54, 1.
6. $\therefore CD \perp$ plane MN. Q.E.D.	6. § 619.

Ex. 1168. In the accompanying diagram AB and CD lie in the same plane. Angle $CBA = 35°$, angle $BCD = 35°$, angle $ABE = 90°$, BE lying in plane MN. Is CD necessarily perpendicular to plane MN? Prove your answer.



Ex. 1169. Can a line be perpendicular to each of two intersecting planes? Prove.

Ex. 1170. If one of two planes is perpendicular to a given line, but the other is not, the planes are not parallel.

Ex. 1171. If a straight line and a plane are each perpendicular to the same straight line, they are parallel to each other.

Proposition XI. Problem

637. *Through a given point to construct a line perpendicular to a given plane.*

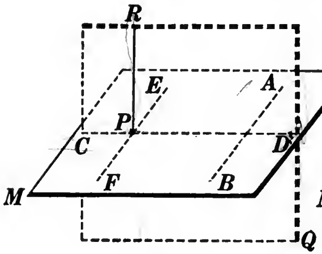


Fig. 1.

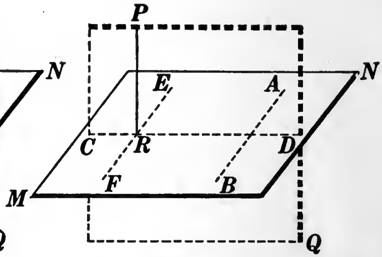


Fig. 2.

Given point P and plane MN.

To construct, through P, a line $\perp$ plane MN.

I. Construction

1. In plane MN draw any convenient line, as AB.
2. Through P construct plane $PQ \perp AB$. § 624.
3. Let plane PQ intersect plane MN in CD. § 616.
4. In plane PQ construct a line through $P \perp CD$, as PR. §§ 148, 149.
5. PR is the perpendicular required.

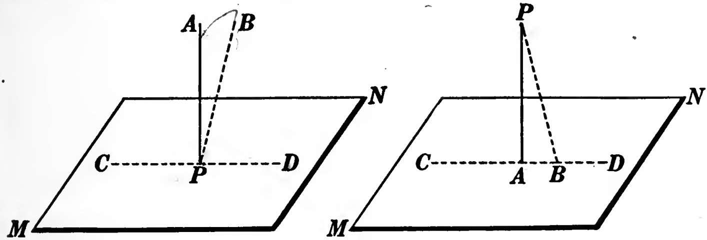
II. Proof

Argument	Reasons
1. Through the foot of PR (P in Fig. 1, R in Fig. 2) in plane MN, draw $EF \parallel AB$.	1. § 179.
2. $AB \perp$ plane PQ.	2. By cons.
3. $\therefore EF \perp$ plane PQ.	3. § 636.
4. $\therefore EF \perp PR$; *i.e.* $PR \perp EF$.	4. § 619.
5. But $PR \perp CD$.	5. By cons.
6. $\therefore PR \perp$ plane MN. Q.E.D.	6. § 622.

III. The discussion will be given in § 639.

Proposition XII. Theorem

638. *Through a given point there exists only one line perpendicular to a given plane.*



Given point P and line PA, through P, $\perp$ plane MN.

To prove PA the only line through $P \perp MN$.

Argument	Reasons
1. Either PA is the only line through $P \perp MN$ or it is not.	1. § 161, *a*.
2. Suppose there exists another line through $P \perp MN$, as PB; then PA and PB determine a plane.	2. § 612.
3. Let this plane intersect plane MN in line CD.	3. § 616.
4. Then PA and PB, two lines through P and lying in the same plane, are $\perp CD$.	4. § 619.
5. This is impossible.	5. §§ 62, 154.
6. $\therefore PA$ is the only line through $P \perp MN$. Q.E.D.	6. § 161, *b*.

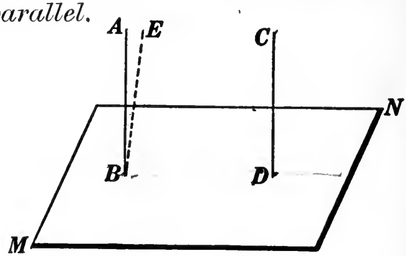
639. §§ 637 and 638 may be combined in one statement as follows:

Through a given point there exists one and only one line perpendicular to a given plane.

Ex. 1172. Find the locus of all points in a plane that are equidistant from two given points not lying in the plane.

Proposition XIII. Theorem

640. *Two straight lines perpendicular to the same plane are parallel.*



Given str. lines AB and $CD \perp$ plane MN.

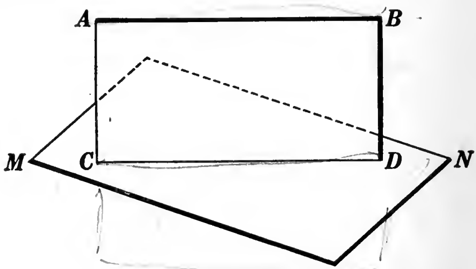
To prove $AB \parallel CD$.

The proof is left as an exercise for the student.

Hint. Suppose that AB is not $\parallel$ CD, but that some other line through B, as BE, is $\parallel$ CD. Use § 638.

Proposition XIV. Theorem

641. *If a straight line is parallel to a plane, the intersection of the plane with any plane passing through the given line is parallel to the given line.*



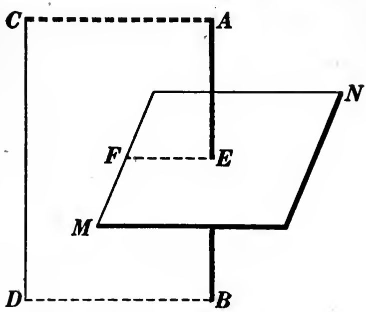
Given line $AB \parallel$ plane MN, and plane AD, through AB, intersecting plane MN in line CD.

To prove $AB \parallel CD$.

The proof is left as an exercise for the student.

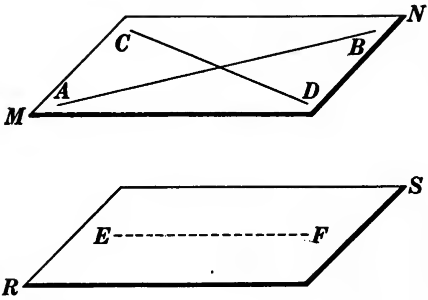
Hint. Suppose that AB is not $\parallel$ CD. Show that AB will then meet plane MN.

642. Cor. I. *If a plane intersects one of two parallel lines, it must, if sufficiently extended, intersect the other also.*



HINT. Pass a plane through AB and CD and let it intersect plane MN in EF. Now if MN does not intersect CD, but is $\parallel$ to it, then $EF \parallel CD$, § 641. Apply § 178.

643. Cor. II. *If two intersecting lines are each parallel to a given plane, the plane of these lines is parallel to the given plane.*



HINT. If plane MN, determined by AB and CD, is not $\parallel$ to plane RS, it will intersect it in some line, as EF. What is the relation of EF to AB and CD?

644. Cor. III. Problem. *Through a given point to construct:*

(*a*) *A line parallel to a given plane.*

(*b*) *A plane parallel to a given plane.*

HINT. (*a*) Let A be a point outside of plane MN. Through A construct any plane intersecting plane MN in line CD. Complete the construction.

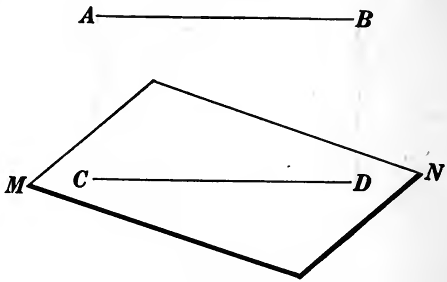
645. Cor. IV. *If two angles, not in the same plane, have their sides parallel respectively, their planes are parallel.*

Ex. 1173. Hold a pointer parallel to the blackboard. Is its shadow on the blackboard parallel to the pointer? why?

Ex. 1174. Find the locus of all straight lines passing through a given point and parallel to a given plane.

Proposition XV. Theorem

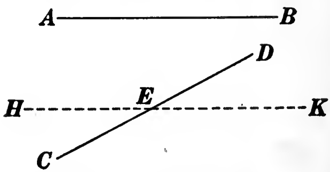
646. *If two straight lines are parallel, a plane containing one of the lines, and only one, is parallel to the other.*



Given ∥ lines AB and CD, and plane MN containing CD.
To prove plane MN ∥ AB.

Argument	Reasons
1. Either plane MN is ∥ AB or it is not.	1. § 161, *a*.
2. Suppose MN is not ∥ AB; then plane MN will intersect AB.	2. § 629.
3. Then plane MN must also intersect CD.	3. § 642.
4. This is impossible, for MN contains CD.	4. By hyp.
5. ∴ plane MN ∥ AB. Q.E.D.	5. § 161, *b*.

647. Cor. I. Problem. *Through a given line to construct a plane parallel to another given line.*

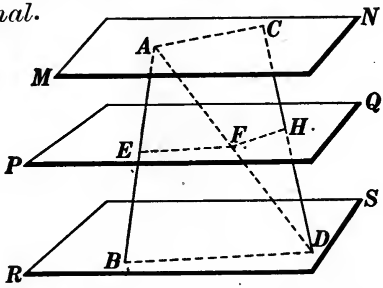


Hint. Through E, any point in CD, construct a line HK ∥ AB.

648. Cor. II. Problem. *Through a given point to construct a plane parallel to any two given straight lines in space.*

PROPOSITION XVI. THEOREM

649. *If two straight lines are intersected by three parallel planes, the corresponding segments of these lines are proportional.*



Given $\parallel$ planes MN, PQ, and RS intersecting line AB in A, E, B and line CD in C, H, D, respectively.

To prove $\dfrac{AE}{EB} = \dfrac{CH}{HD}$.

ARGUMENT	REASONS
1. Draw AD intersecting plane PQ in F.	1. § 54, 15.
2. Let the plane determined by AB and AD intersect PQ in EF and RS in BD.	2. §§ 612, 616.
3. Let the plane determined by AD and DC intersect PQ in FH and MN in AC.	3. §§ 612, 616.
4. $\therefore EF \parallel BD$ and $FH \parallel AC$.	4. § 633.
5. $\therefore \dfrac{AE}{EB} = \dfrac{AF}{FD}$ and $\dfrac{CH}{HD} = \dfrac{AF}{FD}$.	5. § 410.
6. $\therefore \dfrac{AE}{EB} = \dfrac{CH}{HD}$. Q.E.D.	6. § 54, 1.

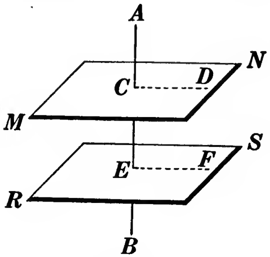
650. Cor. *If two straight lines are intersected by three parallel planes, the lines are divided proportionally.*

Ex. 1175. If any number of lines passing through a common point are cut by two or more parallel planes, their corresponding segments are proportional.

Ex. 1176. In the figure for Prop. XVI, $AE = 6$, $EB = 8$, $AD = 21$, $CD = 28$. Find AF and HD.

Proposition XVII. Theorem

651. *A straight line perpendicular to one of two parallel planes is perpendicular to the other also.*



Given plane *MN* ∥ plane *RS* and line *AB* ⊥ plane *RS*.
To prove line *AB* ⊥ plane *MN*.

Argument Only

1. In plane *MN*, through *C*, draw any line *CD*, and let the plane determined by *AC* and *CD* intersect plane *RS* in *EF*.
2. Then *CD* ∥ *EF*.
3. But *AB* ⊥ *EF*.
4. ∴ *AB* ⊥ *CD*, any line in plane *MN* passing through *C*.
5. ∴ line *AB* ⊥ plane *MN*. Q.E.D.

652. Cor. I. *Through a given point there exists only one plane parallel to a given plane.* (Hint. Apply §§ 638, 644*b*, 651, 625.)

653. §§ 644*b* and 652 may be combined in one statement: *Through a given point there exists one and only one plane parallel to a given plane.*

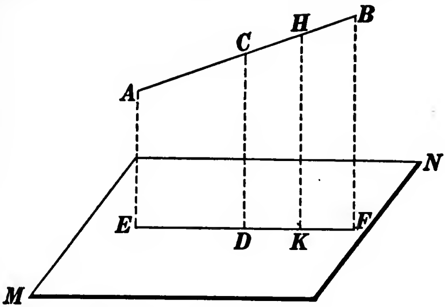
654. Cor. II. *If two planes are each parallel to a third plane, they are parallel to each other.* (Hint. See § 180.)

655. Def. The **projection of a point** upon a plane is the foot of the perpendicular from the point to the plane.

656. Def. The **projection of a line** upon a plane is the locus of the projections of all points of the line upon the plane.

PROPOSITION XVIII. THEOREM

657. *The projection upon a plane of a straight line not perpendicular to the plane is a straight line.*



Given str. line AB not $\perp$ plane MN.

To prove the projection of AB upon MN a str. line.

ARGUMENT	REASONS
1. Through C, any point in AB, draw $CD \perp$ plane MN.	1. § 639.
2. Let the plane determined by AB and CD intersect plane MN in the str. line EF.	2. §§ 612, 616.
3. From H, any point in AB, draw HK, in plane AF, $\parallel CD$.	3. § 179.
4. Then $HK \perp$ plane MN.	4. § 636.
5. $\therefore$ K is the projection of H upon plane MN.	5. § 655.
6. $\therefore$ EF is the projection of AB upon plane MN.	6. § 656.
7. $\therefore$ the projection of AB upon plane MN is a str. line. Q.E.D.	7. Args. 2 and 6.

Ex. 1177. Compare the length of the projection of a line upon a plane with the length of the line itself:

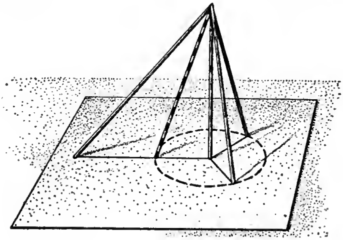
(*a*) If the line is parallel to the plane.
(*b*) If the line is neither parallel nor perpendicular to the plane.
(*c*) If the line is perpendicular to the plane.

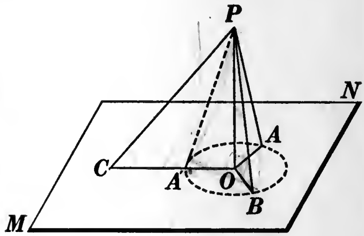
Proposition XIX. Theorem

658. *Of all oblique lines drawn from a point to a plane:*

I. *Those having equal projections are equal.*

II. *Those having unequal projections are unequal, and the one having the greater projection is the longer.*





Given line $PO \perp$ plane MN and:

I. Oblique lines PA and PB with projection $OA =$ projection OB.

II. Oblique lines PA and PC with projection $OC >$ projection OA.

To prove: I. $PB = PA$; II. $PC > PA$.

The proof is left as an exercise for the student.

659. Cor. I. (Converse of Prop. XIX). *Of all oblique lines drawn from a point to a plane:*

I. *Equal oblique lines have equal projections.*

II. *Unequal oblique lines have unequal projections, and the longer line has the greater projection.*

660. Cor. II. *The locus of a point in space equidistant from all points in the circumference of a circle is a straight line perpendicular to the plane of the circle and passing through its center.*

661. Cor. III. *The shortest line from a point to a given plane is the perpendicular from that point to the plane.*

662. Def. The **distance from a point to a plane** is the length of the perpendicular from the point to the plane.

663. Cor. IV. *Two parallel planes are everywhere equally distant.* (Hint. See § 634.)

664. Cor. V. *If a line is parallel to a plane, all points of the line are equally distant from the plane.*

Ex. 1178. In the figure of § 658, if $PO = 12$ inches, $PA = 15$ inches, and $PC = 20$ inches, find OA and CA'.

Ex. 1179. Find the locus of all points in a given plane which are at a given distance from a point outside of the plane.

Ex. 1180. By applying § 660, suggest a practical method of constructing a line perpendicular to a plane:

(*a*) Through a point in the plane;

(*b*) Through a point not in the plane.

Ex. 1181. Find a point in a plane equidistant from all points in the circumference of a circle not lying in the plane.

Ex. 1182. Find the locus of all points equidistant from two parallel planes.

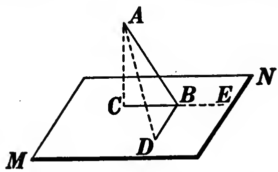
Ex. 1183. Find the locus of all points at a given distance d from a given plane MN.

Ex. 1184. Find the locus of all points in space equidistant from two parallel planes and equidistant from two fixed points.

Ex. 1185. A line and its projection upon a plane always lie in the same plane.

Ex. 1186. (*a*) The acute angle that a straight line makes with its own projection upon a plane is the least angle that it makes with any line passing through its foot in the plane.

(*b*) With what line passing through its foot and lying in the plane does it make the greatest angle?



Hint. (*a*) Measure off $BD = BC$. Which is greater, AD or AC? By means of § 173, prove $\angle ABC < \angle ABD$.

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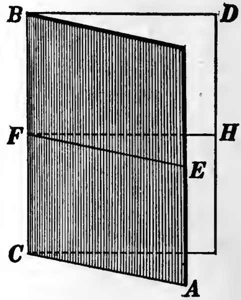
665. Def. The acute angle that a straight line, not perpendicular to a given plane, makes with its own projection upon the plane, is called the **inclination of the line to the plane.**

Ex. 1187. Find the projection of a line 12 inches long upon a plane, if the inclination of the line to the plane is 30°; 45°; 60°.

DIHEDRAL ANGLES

666. Defs. A **dihedral angle** is the figure formed by two planes that diverge from a line. The planes forming a dihedral angle are called its **faces**, and the intersection of these planes, its **edge**.

667. A dihedral angle may be **designated** by reading in order the two planes forming the angle; thus, an angle formed by planes AB and CD is angle $AB–CD$, and is usually written angle $A–BC–D$. If there is no other dihedral angle having the same edge, the line forming the edge is a sufficient designation, as dihedral angle BC.



668. Def. Points, lines, or planes lying in the same plane are said to be **coplanar.**

669. A clear notion of the *magnitude* of a dihedral angle may be obtained by imagining that its two faces, considered as finite portions of planes, were at first coplanar and that one of them has *revolved* about a line common to the two. Thus in the figure we may imagine face CD first to have been in the position of face AB and then to have revolved about BC as an axis to the position of face CD.

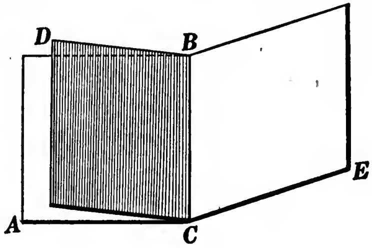
670. Def. The **plane angle of a dihedral angle** is the angle formed by two straight lines, one in each face of the dihedral angle, perpendicular to its edge at the same point. Thus if EF, in face AB, is $\perp$ BC at F, and FH, in face CD, is $\perp$ BC at F, then $\angle EFH$ is the plane-$\angle$ of the dihedral $\angle A–BC–D$.

Ex. 1188. All plane angles of a dihedral angle are equal.

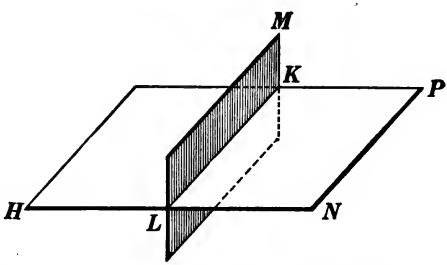
Ex. 1189. Is the plane of angle EFH (§ 667) perpendicular to the edge BC? Prove. State your result in the form of a theorem.

Ex. 1190. Is Ex. 309 true if the quadrilateral is a quadrilateral in space, *i.e.* if the vertices of the quadrilateral are not all in the same plane? Prove.

671. Def. Two dihedral angles are **adjacent** if they have a common edge and a common face which lies between them; thus $\angle A\text{–}BC\text{–}D$ and $\angle D\text{–}CB\text{–}E$ are adj. dihedral $\angle$s.



672. Def. If one plane meets another so as to make two adjacent dihedral angles equal, each of these angles is a **right dihedral angle**, and the planes are said to be **perpendicular** to each other. Thus if plane HP meets plane LM so that dihedral $\angle$s $H\text{–}KL\text{–}M$ and $M\text{–}LK\text{–}N$ are equal, each $\angle$ is a rt. dihedral $\angle$, and planes HP and LM are $\perp$ to each other.



Ex. 1191. By comparison with the definitions of the corresponding terms in plane geometry, frame exact definitions of the following terms: acute dihedral angle; obtuse dihedral angle; reflex dihedral angle; oblique dihedral angle; vertical dihedral angles; complementary dihedral angles; supplementary dihedral angles; bisector of a dihedral angle; alternate interior dihedral angles; corresponding dihedral angles. Illustrate as many of these as you can with an open book.

Ex. 1192. If one plane meets another plane, the sum of the two adjacent dihedral angles is two right dihedral angles.

HINT. See proof of § 65.

Ex. 1193. If the sum of two adjacent dihedral angles is equal to two right dihedral angles, their exterior faces are coplanar.

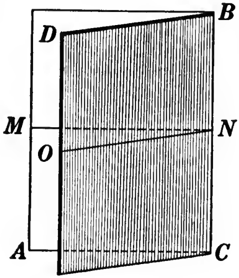
HINT. See proof of § 76.

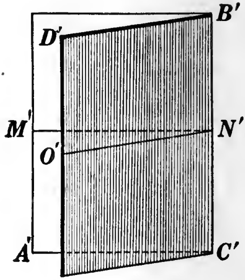
Ex. 1194. If two planes intersect, the vertical dihedral angles are equal.

HINT. See proof of § 77.

Proposition XX. Theorem

673. *If two dihedral angles are equal, their plane angles are equal.*





Given two equal dihedral $\angle$s BC and $B'C'$ whose plane $\angle$s are $\angle$s MNO and $M'N'O'$, respectively.

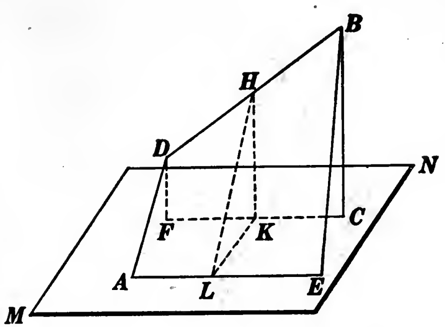
To prove $\angle MNO = \angle M'N'O'$.

Argument	Reasons
1. Superpose dihedral $\angle BC$ upon its equal, dihedral $\angle B'C'$, so that point N of edge BC shall fall upon point N' of edge $B'C'$.	1. § 54, 14.
2. Then MN and $M'N'$, two lines in plane AB, are $\perp BC$ at point N.	2. § 670.
3. $\therefore MN$ and $M'N'$ are collinear.	3. § 62.
4. Likewise NO and $N'O'$ are collinear.	4. §§ 670, 62.
5. $\therefore \angle MNO = \angle M'N'O'$. Q.E.D.	5. § 18.

674. Cor. I. *The plane angle of a right dihedral angle is a right angle.*

675. Cor. II. *If two intersecting planes are each perpendicular to a third plane, their intersections with the third plane intersect each other.*

Given planes AB and $CD \perp$ plane MN and intersecting each other in line DB; also let AE and FC be the intersections of planes AB and CD with plane MN.



To prove that AE and FC intersect each other.

ARGUMENT	REASONS
1. Either $AE \parallel FC$ or AE and FC intersect each other.	1. 161, *a*.
2. Suppose $AE \parallel FC$. Then through H, any point in DB, pass a plane $HKL \perp FC$, intersecting FC in K and AE in L.	2. § 627.
3. Then plane HKL is $\perp AE$ also.	3. § 636.
4. $\therefore \angle HKL$ is the plane $\angle$ of dihedral $\angle FC$, and $\angle KLH$ is the plane $\angle$ of dihedral $\angle AE$.	4. § 670.
5. But dihedral $\angle$s FC and AE are rt. dihedral $\angle$s.	5. § 672.
6. $\therefore \angle$s HKL and KLH are rt. $\angle$s.	6. § 674.
7. $\therefore \triangle HKL$ contains two rt. $\angle$s.	7. Arg. 6.
8. But this is impossible.	8. § 206.
9. $\therefore AE$ and FC intersect each other. Q.E.D.	9. § 161, *b*.

Ex. 1195. Find the locus of all points equidistant from two given points in space.

Ex. 1196. Find the locus of all points equidistant from three given points in space.

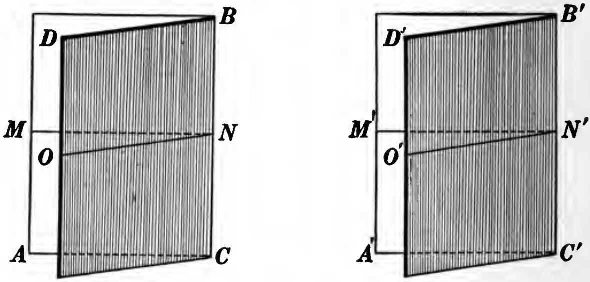
Ex. 1197. Are the supplements of equal dihedral angles equal? complements? Prove your answer.

Ex. 1198. If two planes are each perpendicular to a third plane, can they be parallel to each other? Explain. If they are parallel to each other, prove their intersections with the third plane parallel.

Proposition XXI. Theorem

(Converse of Prop. XX)

676. *If the plane angles of two dihedral angles are equal, the dihedral angles are equal.*



Given two dihedral $\angle s$ BC and $B'C'$ whose plane $\angle s$ MNO and $M'N'O'$ are equal.

To prove dihedral $\angle BC =$ dihedral $\angle B'C'$.

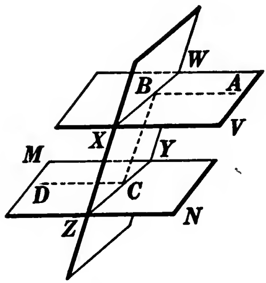
Argument	Reasons
1. Place dihedral $\angle BC$ upon dihedral $\angle B'C'$ so that plane $\angle MNO$ shall be superposed upon its equal, plane $\angle M'N'O'$.	1. § 54, 14.
2. Then BC and $B'C'$ are both $\perp$ MN and NO at N.	2. § 670.
3. $\therefore$ BC and $B'C'$ are both $\perp$ plane MNO at N.	3. § 622.
4. $\therefore$ BC and $B'C'$ are collinear.	4. § 638.
5. $\therefore$ planes AB and $A'B'$, determined by MN and BC, are coplanar; also planes CD and $C'D'$, determined by BC and NO, are coplanar.	5. § 612.
6. $\therefore$ dihedral $\angle BC =$ dihedral $\angle B'C'$.	6. § 18.

Q.E.D.

677. **Cor.** *If the plane angle of a dihedral angle is a right angle, the dihedral angle is a right dihedral angle.*

Ex. 1199. Prove Ex. 1194 by applying § 676.

Ex. 1200. If two parallel planes are cut by a transversal plane, the alternate interior dihedral angles are equal.



HINT. Let $\angle ABC$ be the plane $\angle$ of dihedral $\angle V\text{-}WX\text{-}Y$. Let the plane determined by AB and BC intersect plane MN in CD. Then AB and CD lie in the same plane and are $\parallel$ (§ 633). Prove that $\angle DCB$ is the plane $\angle$ of dihedral $\angle M\text{-}ZY\text{-}X$.

Ex. 1201. State the converse of Ex. 1200, and prove it by the indirect method.

Ex. 1202. If two parallel planes are cut by a transversal plane, the corresponding dihedral angles are equal. (HINT. See proof of § 190.)

Ex. 1203. State the converse of Ex. 1202, and prove it by the indirect method.

Ex. 1204. If two parallel planes are cut by a transversal plane, the sum of the two interior dihedral angles on the same side of the transversal plane is two right dihedral angles. (HINT. See proof of § 192.)

Ex. 1205. Two dihedral angles whose faces are parallel, each to each, are either equal or supplementary dihedral angles. (HINT. See proof of § 198.)

Ex. 1206. A dihedral angle has the same numerical measure as its plane angle. (HINT. Proof similar to that of § 358.)

Ex. 1207. Two dihedral angles have the same ratio as their plane angles.

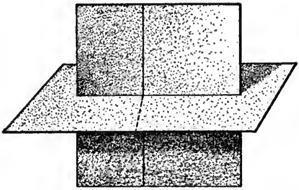
Ex. 1208. Find a point in a plane equidistant from three given points not lying in the plane.

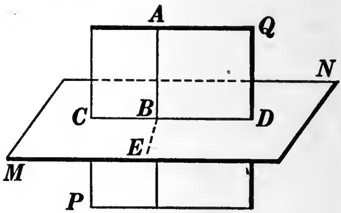
Ex. 1209. If a straight line intersects one of two parallel planes, it must, if sufficiently prolonged, intersect the other also. (HINT. Use the indirect method and apply §§ 663 and 664.)

Ex. 1210. If a plane intersects one of two parallel planes, it must, if sufficiently extended, intersect the other also. (HINT. Use the indirect method and apply § 652.)

PROPOSITION XXII. THEOREM

678. *If a straight line is perpendicular to a plane, every plane containing this line is perpendicular to the given plane.*



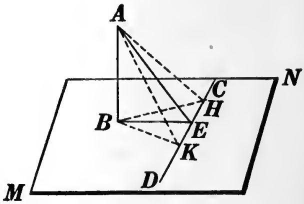


Given str. line $AB \perp$ plane MN and plane PQ containing line AB and intersecting plane MN in CD.

To prove plane $PQ \perp$ plane MN.

ARGUMENT	REASONS
1. $AB \perp CD$.	1. § 619.
2. Through B, in plane MN, draw $BE \perp CD$.	2. § 63.
3. Then $\angle ABE$ is the plane $\angle$ of dihedral $\angle Q\text{–}CD\text{–}M$.	3. § 670.
4. But $\angle ABE$ is a rt. $\angle$.	4. § 619.
5. $\therefore$ dihedral $\angle Q\text{–}CD\text{–}M$ is a rt. dihedral $\angle$, and plane $PQ \perp$ plane MN. Q.E.D.	5. § 677.

Ex. 1211. If from the foot of a perpendicular to a plane a line is drawn at right angles to any line in the plane, the line drawn from the point of intersection so formed to any point in the perpendicular is perpendicular to the line of the plane.

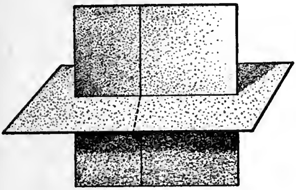


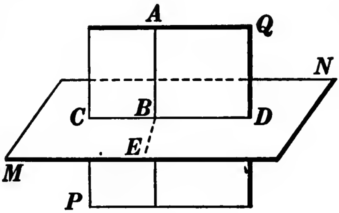
HINT. Make $KE = EH$. Prove $AK = AH$, and apply § 142.

Ex. 1212. In the figure of Ex. 1211, if $AB = 20$, $BE = 4\sqrt{11}$, and $EK = 10$, find AK.

PROPOSITION XXIII. THEOREM

679. *If two planes are perpendicular to each other, any line in one of them, perpendicular to their intersection, is perpendicular to the other.*





Given plane $PQ \perp$ plane MN, CD their line of intersection, and AB, in plane PQ, $\perp CD$.

To prove $AB \perp$ plane MN.

ARGUMENT	REASONS
1. Through B, in plane MN, draw $BE \perp CD$.	1. § 63.
2. Then $\angle ABE$ is the plane $\angle$ of the rt. dihedral $\angle$ $Q\text{–}CD\text{–}M$.	2. § 670.
3. $\therefore \angle ABE$ is a rt. $\angle$, and $AB \perp BE$.	3. § 674.
4. But $AB \perp CD$.	4. By hyp.
5. $\therefore AB \perp$ plane MN. Q.E.D.	5. § 622.

680. Cor. *If two planes are perpendicular to each other, a line perpendicular to one of them at any point in their line of intersection, lies in the other.*

HINT. Apply the indirect method, using §§ 679 and 638.

Ex. 1213. If a plane is perpendicular to the edge of a dihedral angle, is it perpendicular to each of the faces of the dihedral angle? Prove your answer.

Ex. 1214. The plane containing a straight line and its projection upon a plane is perpendicular to the given plane.

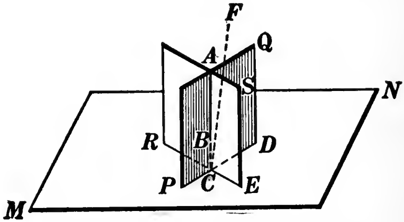
Ex. 1215. If two planes are perpendicular to each other, a line perpendicular to one of them from any point in the other lies in the other plane.

Proposition XXIV. Theorem

681. *If each of two intersecting planes is perpendicular to a third plane:*

I. *Their line of intersection intersects the third plane.*

II. *Their line of intersection is perpendicular to the third plane.*



Given planes PQ and $RS \perp$ plane MN and intersecting each other in line AB.

To prove: I. That AB intersects plane MN.

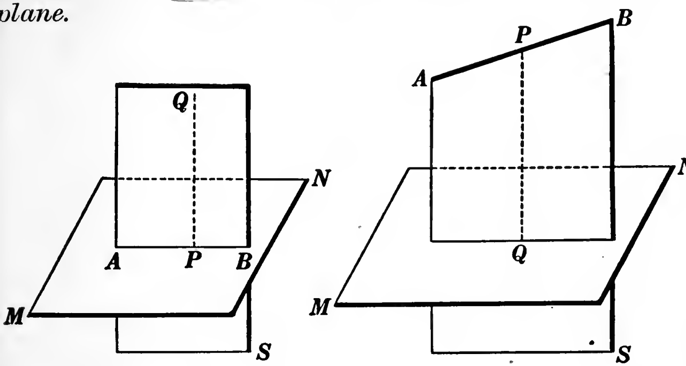
II. $AB \perp$ plane MN.

I. Argument	Reasons
1. Let planes PQ and RS intersect plane MN in lines PD and RE, respectively.	1. § 616.
2. Then PD and RE intersect in a point as C.	2. § 675.
3. ∴ AB passes through C; *i.e.* AB intersects plane MN. Q.E.D.	3. § 617, I.

II. Argument	Reasons
1. Either $AB \perp$ plane MN or it is not.	1. § 161, *a*.
2. Suppose AB is not $\perp$ plane MN, but that some other line through C, the point common to the three planes, is $\perp$ plane MN, as line CF.	2. § 639.
3. Then CF lies in plane PQ, also in plane RS.	3. § 680.
4. ∴ CF is the intersection of planes PQ and RS.	4. § 614.
5. ∴ planes PQ and RS intersect in two str. lines, which is impossible.	5. § 616.
6. ∴ $AB \perp$ plane MN. Q.E.D.	6. § 161, *b*.

PROPOSITION XXV. PROBLEM

682. *Through any straight line, not perpendicular to a plane, to construct a plane perpendicular to the given plane.*



Given line AB not $\perp$ plane MN.

To construct, through AB, a plane $\perp$ plane MN.

The construction, proof, and discussion are left as an exercise for the student.

HINT. Apply § 678. For discussion, see § 683.

683. Cor. *Through a straight line, not perpendicular to a plane, there exists only one plane perpendicular to the given plane.*

HINT. Suppose there should exist another plane through $AB \perp$ plane MN. What would you know about AB?

684. §§ 682 and 683 may be combined in one statement as follows:

Through a straight line, not perpendicular to a plane, there exists one and only one plane perpendicular to the given plane.

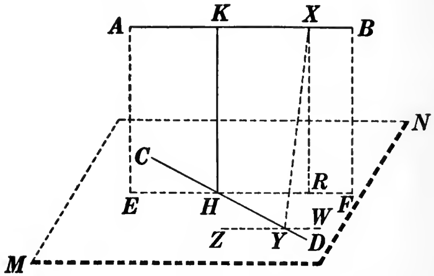
Ex. 1216. Apply the truth of Prop. XXIV: (*a*) to the planes that intersect at the corner of a room; (*b*) to the planes formed by an open book placed perpendicular to the top of the desk.

Ex. 1217. If a plane is perpendicular to each of two intersecting planes, it is perpendicular to their intersection.

PROPOSITION XXVI. PROBLEM

skew

685. *To construct a common perpendicular to any two straight lines in space.*



Given AB and CD, any two str. lines in space.

To construct a line $\perp$ both to AB and to CD.

I. Construction

1. Through CD construct plane $MN \parallel AB$. § 647.
2. Through AB construct plane $AF \perp$ plane MN intersecting MN in EF, and CD in H. § 682.
3. Through H construct HK, in plane AF, $\perp EF$. § 148.
4. HK is $\perp$ to both AB and CD and is the line required.

II. Proof

ARGUMENT	REASONS
1. $AB \parallel$ plane MN.	1. By cons.
2. $\therefore EF \parallel AB$.	2. § 641.
3. But $HK \perp EF$.	3. By cons.
4. $\therefore HK \perp AB$.	4. § 193.
5. Also $HK \perp$ plane MN.	5. § 679.
6. $\therefore HK \perp CD$.	6. § 619.
7. $\therefore HK \perp$ to both AB and CD. Q.E.D.	7. Args. 4 and 6.

III. The discussion will be given in § 686.

686. Cor. *Between two straight lines in space (not in the same plane) there exists only one common perpendicular.*

HINT. Suppose XY, in figure of § 685, a second $\perp$ to AB and CD. Through Y draw $ZW \parallel AB$. What is the relation of XY to AB? to ZW? to CD? to plane MN? Through X draw $XR \perp EF$. What is the relation of XR to plane MN? Complete the proof.

687. §§ 685 and 686 may be combined in one statement as follows:

Between two straight lines in space (not in the same plane) there exists one and only one common perpendicular.

Ex. 1218. A room is 20 feet long, 15 feet wide, and 10 feet high. Find the length of the shortest line that can be drawn on floor and walls from a lower corner to the diagonally opposite corner. Find the length of the line that extends diagonally across the floor, then along the intersection of two walls to the ceiling.

Ex. 1219. If two equal lines are drawn from a given point to a given plane, the inclinations of these lines to the given plane are equal. If two unequal lines are thus drawn, which has the greater inclination? Prove.

Ex. 1220. The two planes determined by two parallel lines and a point not in their plane, intersect in a line which is parallel to each of the given parallels.

Ex. 1221. If two lines are parallel, their projections on a plane are either the same line, or parallel lines.

Ex. 1222. If each of three planes is perpendicular to the other two: (a) the intersection of any two of the planes is perpendicular to the third plane; (b) each of the three lines of intersection is perpendicular to the other two. Find an illustration of this exercise in the classroom.

Ex. 1223. If two planes are parallel, no line in the one can meet any line in the other.

Ex. 1224. Find all points equidistant from two parallel planes and equidistant from three points: (a) if the points lie in neither plane; (b) if the points lie in one of the planes.

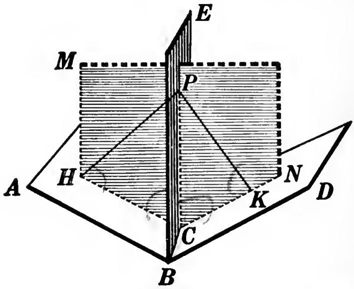
Ex. 1225. Find all points equidistant from two given points, equidistant from two parallel planes, and at a given distance d from a third plane.

Ex. 1226. If each of two intersecting planes is parallel to a given line, the intersection of the planes is parallel to the line.

Ex. 1227. Construct, through a point in space, a straight line that shall be parallel to two intersecting planes.

Proposition XXVII. Theorem

688. *Every point in the plane that bisects a dihedral angle is equidistant from the faces of the angle.*



Given plane BE bisecting the dihedral $\angle$ formed by planes AC and CD; also PH and PK $\perp$s from P, any point in plane BE, to faces AC and CD, respectively.

To prove $PH = PK$.

Argument	Reasons
1. Through PH and PK pass plane MN intersecting plane AC in CH, plane CD in CK, plane BE in PC, and edge BC in C.	1. § 612, 616.
2. Then plane $MN \perp$ planes AC and CD; *i.e.* planes AC and CD are $\perp$ plane MN.	2. § 678.
3. $\therefore$ $BC \perp$ plane MN.	3. § 681, II.
4. $\therefore$ $BC \perp CH$, CP, and CK.	4. § 619.
5. $\therefore$ $\angle$s PCH and KCP are the plane $\angle$s of the dihedral $\angle$s $E\text{–}BC\text{–}A$ and $D\text{–}CB\text{–}E$.	5. § 670.
6. But dihedral $\angle E\text{–}BC\text{–}A$ = dihedral $\angle D\text{–}CB\text{–}E$.	6. By hyp.
7. $\therefore \angle PCH = \angle KCP$.	7. § 673.
8. Also $PC = PC$.	8. By iden.
9. $\therefore$ rt. $\triangle PCH =$ rt. $\triangle KCP$.	9. § 209.
10. $\therefore PH = PK$. Q.E.D.	10. § 110.

689. Cor. I. *Every point equidistant from the two faces of a dihedral angle lies in the plane bisecting the angle.*

690. Cor. II. *The plane bisecting a dihedral angle is the locus of all points in space equidistant from the faces of the angle.*

691. Cor. III. Problem. *To construct the bisector of a given dihedral angle.*

Ex. 1228. Prove that a dihedral angle can be bisected by only one plane.

Hint. See proof of § 599.

Ex. 1229. Find the locus of all points equidistant from two intersecting planes. Of how many planes does this locus consist?

Ex. 1230. Find the locus of all points in space equidistant from two intersecting lines. Of how many planes does this locus consist?

Ex. 1231. Find the locus of all points in space equidistant from two parallel lines.

Ex. 1232. Find the locus of all points in space equidistant from two intersecting planes and equidistant from all points in the circumference of a circle.

Ex. 1233. Find the locus of all points in space equidistant from two intersecting planes and equidistant from two fixed points.

Ex. 1234. Find the locus of all points in space equidistant from two intersecting planes, equidistant from two parallel planes, and equidistant from two fixed points.

Ex. 1235. If from any point within a dihedral angle lines are drawn perpendicular to the faces of the angle, the angle formed by the perpendiculars is supplementary to the plane angle of the dihedral angle.

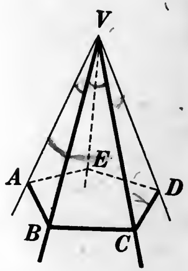
Ex. 1236. Given two points, P and Q, one in each of two intersecting planes, M and N. Find a point X in the intersection of planes M and N such that $PX+XQ$ is a minimum.

Ex. 1237. Given two points, P and Q, on one side of a given plane MN. Find a point X in plane MN such that $PX+XQ$ shall be a minimum.

Hint. See Ex. 175.

POLYHEDRAL ANGLES

692. Def. A **polyhedral angle** is the figure generated by a moving straight line segment that continually intersects the boundary of a fixed polygon and one extremity of which is a fixed point not in the plane of the given polygon. A polyhedral angle is sometimes called a **solid angle.**



693. Defs. The moving line is called the **generatrix,** as *VA*; the fixed polygon is called the **directrix,** as polygon *ABCDE*; the fixed point is called the **vertex** of the polyhedral angle, as *V*.

694. Defs. The generatrix in any position is an **element** of the polyhedral angle; the elements through the vertices of the polygon are the **edges,** as *VA*, *VB*, etc.; the portions of the planes determined by the edges of the polyhedral angle, and limited by them are the **faces,** as *AVB*, *BVC*, etc.; the angles formed by the edges are the **face angles,** as ∠s *AVB*, *BVC*, etc.; the dihedral angles formed by the faces are called the **dihedral angles** of the polyhedral angle, as dihedral ∠s *VA*, *VB*, etc.

695. Def. The face angles and the dihedral angles taken together are sometimes called the **parts of a polyhedral angle.**

696. A polyhedral angle may be **designated** by a letter at the vertex and one on each edge, as *V–ABCDE*. If there is no other polyhedral angle having the same vertex, the letter at the vertex is a sufficient designation, as *V*.

697. Def. A **convex polyhedral angle** is a polyhedral angle whose directrix is a convex polygon, *i.e.* a polygon no side of which, if prolonged, will enter the polygon; as *V–ABCDE*. In this text only convex polyhedral angles will be considered.

698. Defs. A **trihedral angle** is a polyhedral angle whose directrix is a triangle (*tri*-gon); a **tetrahedral angle,** a polyhedral angle whose directrix is a quadrilateral (*tetra*-gon); etc.

699. Defs. A trihedral angle is called a **rectangular, birectangular,** or **trirectangular** trihedral angle according as it contains one, two, or three right dihedral angles.

700. Def. An **isosceles trihedral angle** is a trihedral angle having two face angles equal.

Ex. 1238. By holding an open book perpendicular to the desk, illustrate birectangular and trirectangular trihedral angles. By placing one face of the open book on top of the desk and the other face along the side of the desk against the edge, illustrate a rectangular trihedral angle.

Ex. 1239. Is every birectangular trihedral angle isosceles? Is every isosceles trihedral angle birectangular?

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701. From the general definition of equal geometric figures (§ 18) it follows that:

Two polyhedral angles are equal if they can be made to coincide.

PROPOSITION XXVIII. THEOREM

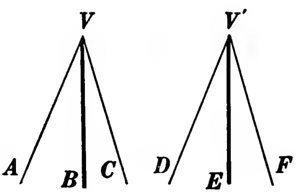
702. *Two trihedral angles are equal:*

I. *If a face angle and the two adjacent dihedral angles of one are equal respectively to a face angle and the two adjacent dihedral angles of the other;*

II. *If two face angles and the included dihedral angle of one are equal respectively to two face angles and the included dihedral angle of the other:*

provided the equal parts are arranged in the same order.

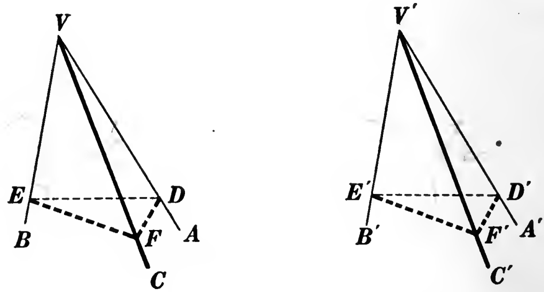
The proofs are left as exercises for the student.



703. Questions. Compare carefully the wording of I above and the accompanying figures with the wording and figures of § 105. What in I takes the place of △ in § 105? side? adj. ∠s? What, in the accompanying figure, corresponds to △ ABC in the proof of § 105? △ DEF? AC? DF? point A? point C? If these and similar changes are made in the proof of § 105, will it serve as a proof of I above? Compare II above with § 107.

Proposition XXIX. Theorem

704. *Two trihedral angles are equal if the three face angles of one are equal respectively to the three face angles of the other, and the equal parts are arranged in the same order.*



Given trihedral $\measuredangle$s $V\text{–}ABC$ and $V'\text{–}A'B'C'$, $\angle AVB = \angle A'V'B'$, $\angle BVC = \angle B'V'C'$, $\angle CVA = \angle C'V'A'$, and the equal face angles arranged in the same order.

To prove trihedral $\angle V\text{–}ABC =$ trihedral $\angle V'\text{–}A'B'C'$.

Outline of Proof

1. Since, by hyp., any two face $\measuredangle$s of $V\text{–}ABC$, as $\measuredangle$s AVB and BVC, are equal, respectively, to the two corresponding face $\measuredangle$s of $V'\text{–}A'B'C'$, it remains only to prove the included dihedral $\measuredangle$s VB and $V'B'$ equal. § 702, II. (See also § 705.)
2. Let face $\measuredangle$s AVB and BVC be oblique $\measuredangle$s; then from any point E in VB, draw ED and EF, in planes AVB and BVC, respectively, and $\perp VB$.
3. Since $\measuredangle$s AVB and BVC are oblique $\measuredangle$s, ED and EF will meet VA and VC in D and F, respectively. Draw FD.
4. Similarly, lay off $V'E' = VE$ and draw $\triangle D'E'F'$.
5. Prove rt. $\triangle DVE =$ rt. $\triangle D'V'E'$; then $VD = V'D'$, $ED = E'D'$.
6. Prove rt. $\triangle EVF =$ rt. $\triangle E'V'F'$; then $VF = V'F'$, $EF = E'F'$.
7. Prove $\triangle FVD = \triangle F'V'D'$; then $FD = F'D'$.
8. $\therefore \triangle DEF = \triangle D'E'F'$; then $\angle DEF = \angle D'E'F'$.

9. But $\angle$s DEF and $D'E'F'$ are the plane $\angle$s of dihedral $\angle$s VB and $V'B'$, respectively.

10. $\therefore$ dihedral $\angle\ VB =$ dihedral $\angle\ V'B'$.

11. $\therefore$ trihedral $\angle\ V\text{–}ABC =$ trihedral $\angle\ V'\text{–}A'B'C'$. Q.E.D.

705. Note. If all the face $\angle$s are rt. $\angle$s, show that all the dihedral $\angle$s are rt. dihedral $\angle$s and hence that all are equal. If two face $\angle$s of a trihedral $\angle$ are rt. $\angle$s, show that the third face $\angle$ is the plane $\angle$ of the included dihedral $\angle$, and hence that two homologous dihedral $\angle$s, as VB and $V'B'$, are equal. It remains to prove that Prop. XXIX is true if only one face $\angle$ of the first trihedral $\angle$ and its homologous face $\angle$ of the other are rt. $\angle$s, or if all face $\angle$s are oblique.

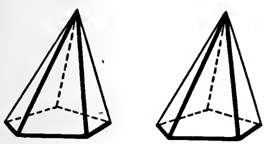
706. Questions. State the proposition in Bk. I that corresponds to § 704. What was the main step in the proof of that proposition? Did that correspond to proving dihedral $\angle\ VB$ of § 704 $=$ dihedral $\angle\ V'B'$?

707. Def. Two polyhedral angles are said to be **symmetrical** if their corresponding parts are equal but arranged in reverse order.

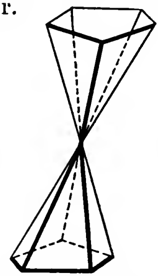
By making symmetrical polyhedral angles and comparing them, the student can easily satisfy himself that in general they cannot be made to coincide.

708. Def. Two polyhedral angles are said to be **vertical** if the edges of each are the prolongations of the edges of the other.

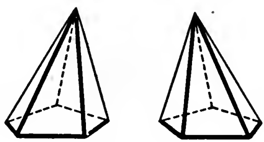
It will be seen that two vertical, like two symmetrical, polyhedral angles have their corresponding parts equal but arranged in reverse order.



Two Equal Polyhedral Angles



Two Vertical Polyhedral Angles



Two Symmetrical Polyhedral Angles

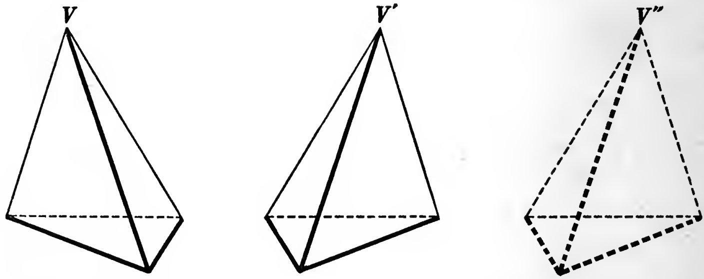
Proposition XXX. Theorem

709. *Two trihedral angles are symmetrical:*

I. *If a face angle and the two adjacent dihedral angles of one are equal respectively to a face angle and the two adjacent dihedral angles of the other;*

II. *If two face angles and the included dihedral angle of one are equal respectively to two face angles and the included dihedral angle of the other;*

III. *If the three face angles of one are equal respectively to the three face angles of the other:*
provided the equal parts are arranged in reverse order.



The proofs are left as exercises for the student.

Hint. Let V and V' be the two trihedral $\measuredangle$ with parts equal but arranged in reverse order. Construct trihedral $\angle V''$ symmetrical to V. Then what will be the relation of V'' to V'? of V' to V?

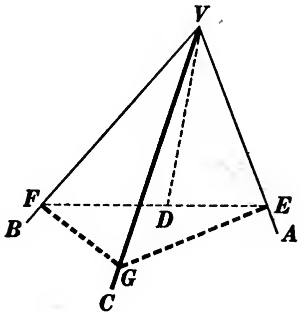
Ex. 1240. Can two polyhedral angles be symmetrical and equal? vertical and equal? symmetrical and vertical? If two polyhedral angles are vertical, are they necessarily symmetrical? if symmetrical, are they necessarily vertical?

Ex. 1241. Are two trirectangular trihedral angles necessarily equal? Are two birectangular trihedral angles equal? Prove your answers.

Ex. 1242. If two trihedral angles have three face angles of one equal respectively to three face angles of the other, the dihedral angles of the first are equal respectively to the dihedral angles of the second.

Proposition XXXI. Theorem

710. *The sum of any two face angles of a trihedral angle is greater than the third face angle.*



Given trihedral $\angle\ V\text{-}ABC$ in which the greatest face $\angle$ is AVB.

To prove $\angle\ BVC + \angle\ CVA > \angle\ AVB$.

Outline of Proof

1. In face AVB draw VD making $\angle\ DVB = \angle\ BVC$, and through D draw any line intersecting VA in E and VB in F.
2. On VC lay off $VG = VD$ and draw FG and GE.
3. Prove $\triangle\ FVG = \triangle\ DVF$; then $FG = FD$.
4. But $FG + GE > FD + DE$; $\therefore\ GE > DE$.
5. In $\triangle$s GVE and EVD, prove $\angle\ GVE > \angle\ EVD$.
6. But $\angle\ FVG = \angle\ DVF$.
7. $\therefore\ \angle\ FVG + \angle\ GVE > \angle\ EVD + \angle\ DVF$;
   i.e. $\angle\ BVC + \angle\ CVA > \angle\ AVB$. Q.E.D.

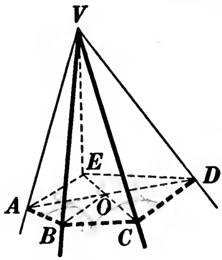
711. Question. State the theorem in Bk. I that corresponds to Prop. XXXI. Can that theorem be proved by a method similar to the one used here? If so, give the proof.

Ex. 1243. If, in trihedral angle $V\text{-}ABC$, angle $BVC = 60°$, and angle $CVA = 80°$, make a statement as to the number of degrees in angle AVB.

Ex. 1244. Any face angle of a trihedral angle is greater than the difference of the other two.

Proposition XXXII. Theorem

712. *The sum of all the face angles of any convex polyhedral angle is less than four right angles.*



Given polyhedral $\angle V$ with n faces.

To prove the sum of the face $\angle$s at $V < 4$ rt. $\angle$s.

Hint. Let a plane intersect the edges of the polyhedral $\angle$ in A, B, C, etc. From O, any point in polygon ABC . . ., draw OA, OB, OC, etc. How many $\triangle$s have their vertices at V? at O? What is the sum of all the $\angle$s of all the $\triangle$s with vertices at V? at O? Which is the greater, $\angle ABV + \angle VBC$ or $\angle ABO + \angle OBC$? Then which is the greater, the sum of the base $\angle$s of $\triangle$s with vertices at V, or the sum of the base $\angle$s of $\triangle$s with vertices at O? Then which is greater, the sum of the face $\angle$s about V, or the sum of the $\angle$s about O?

713. Question. Is there a proposition in plane geometry corresponding to Prop. XXXII? If so, state it. If not, state the one that most nearly corresponds to it.

Ex. 1245. Can a polyhedral angle have for its faces three equilateral triangles? four? five? six?

Ex. 1246. Can a polyhedral angle have for its faces three squares? four?

Ex. 1247. Can a polyhedral angle have for its faces three regular pentagons? four?

Ex. 1248. Show that the greatest number of polyhedral angles that can possibly be formed with regular polygons as faces is five.

Ex. 1249. Can a trihedral angle have for its faces a regular decagon and two equilateral triangles? a regular decagon, an equilateral triangle, and a square? two regular octagons and a square?

BOOK VII

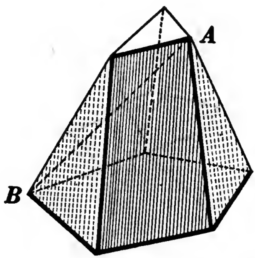
POLYHEDRONS

714. Def. A surface is said to be **closed** if it separates a finite portion of space from the remaining space.

715. Def. A **solid closed figure** is a figure in space composed of a closed surface and the finite portion of space bounded by it.

716. Def. A **polyhedron** is a solid closed figure whose bounding surface is composed of planes only.

717. Defs. The intersections of the bounding planes are called the **edges**; the intersections of the edges, the **vertices**; and the portions of the bounding planes bounded by the edges, the **faces**, of the polyhedron.



Polyhedron

718. Def. A **diagonal** of a polyhedron is a straight line joining any two vertices not in the same face, as AB.

719. Defs. A polyhedron of four faces is called a **tetrahedron**; one of six faces, a **hexahedron**; one of eight faces, an **octahedron**; one of twelve faces, a **dodecahedron**; one of twenty faces, an **icosahedron**; etc.

Ex. 1250. How many diagonals has a tetrahedron? a hexahedron?

Ex. 1251. What is the least number of faces that a polyhedron can have? edges? vertices?

Ex. 1252. How many edges has a tetrahedron? a hexahedron? an octahedron?

Ex. 1253. How many vertices has a tetrahedron? a hexahedron? an octahedron?

Ex. 1254. If E represents the number of edges, F the number of faces, and V the number of vertices in each of the polyhedrons mentioned in Exs. 1252 and 1253, show that in each case $E + 2 = V + F$. This result is known as Euler's theorem.

Ex. 1255. Show that in a tetrahedron $S = (V-2)$ 4 right angles, where S is the sum of the face angles and V is the number of vertices.

Ex. 1256. Does the formula, $S = (V-2)$ 4 right angles, hold for a hexahedron? an octahedron? a dodecahedron?

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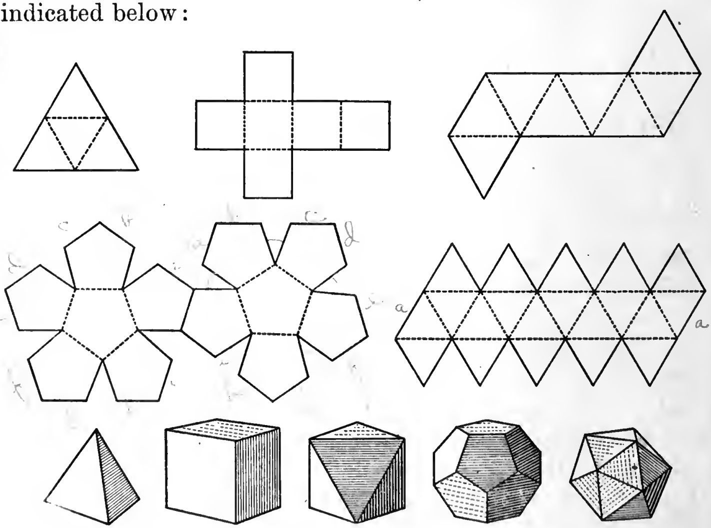
720. Def. A **regular polyhedron** is a polyhedron all of whose faces are equal regular polygons, and all of whose polyhedral angles are equal.

721. Questions. How many equilateral triangles can meet to form a polyhedral angle (§ 712)? Then what is the greatest number of regular polyhedrons possible having equilateral triangles as faces? What is the greatest number of regular polyhedrons possible having squares as faces? having regular pentagons as faces? Can a regular polyhedron have as faces regular polygons of more than five sides? why? What, then, is the maximum number of kinds of regular polyhedrons possible?

722. From the questions in § 721, the student has doubtless drawn the conclusion that *not more* than five kinds of regular polyhedrons exist. He should convince himself that these five *are* possible by actually making them from cardboard as indicated below:

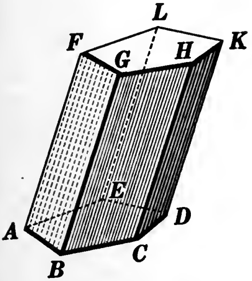


Tetrahedron Hexahedron Octahedron Dodecahedron Icosahedron

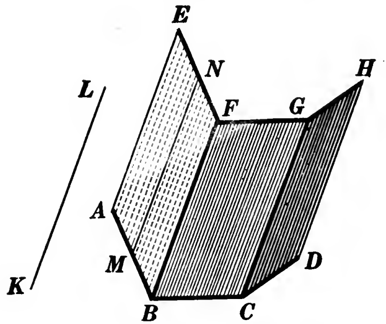
723. Historical Note. The Pythagoreans knew that there were five regular polyhedrons, but it was Euclid who proved that there can be *only five.* Hippasus (*circ.* 470 B.C.), who discovered the dodecahedron, is said to have been drowned for announcing his discovery, as the Pythagoreans were pledged to refer the glory of any new discovery "back to the founder."

PRISMS *

724. Def. A **prismatic surface** is a surface generated by a moving straight line that continually intersects a fixed broken line and remains parallel to a fixed straight line not coplanar with the given broken line.



Prism



Prismatic Surface

725. Defs. By referring to § 693, the student may give the definitions of **generatrix** and **directrix** of a prismatic surface. Point these out in the figure.

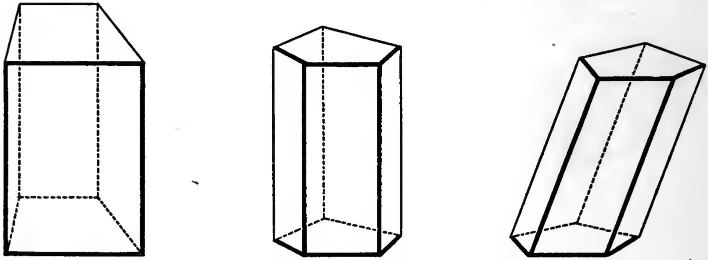
726. Def. A **prism** is a polyhedron whose boundary consists of a prismatic surface and two parallel planes cutting the generatrix in each of its positions.

727. Defs. The two parallel plane sections are the **bases** of the prism, as $ABCDE$ and $FGHKL$; the faces forming the prismatic surface are the **lateral faces,** as AG, BH, etc.; the intersections of the lateral faces are the **lateral edges,** as AF, BG, etc.

In this text only prisms whose bases are convex polygons will be considered.

* This treatment of prisms and pyramids is given because of its similarity to the treatment of cylinders and cones given in §§ 819–822 and 837–840.

728. Def. A **right section** of a prism is a section formed by a plane which is perpendicular to a lateral edge of the prism and which cuts the lateral edges or the edges prolonged.



Right Prism Regular Prism Oblique Prism

729. Def. A **right prism** is a prism whose lateral edges are perpendicular to the bases.

730. Def. A **regular prism** is a right prism whose bases are regular polygons.

731. Def. An **oblique prism** is a prism whose lateral edges are oblique to the bases.

732. Defs. A prism is **triangular, quadrangular,** etc., according as its bases are triangles, quadrilaterals, etc.

733. Def. The **altitude** of a prism is the perpendicular from any point in the plane of one base to the plane of the other base.

734. The following are some of the properties of a prism; the student should prove the correctness of each:

(a) *Any two lateral edges of a prism are parallel.*

(b) *The lateral edges of a prism are equal.*

(c) *Any lateral edge of a right prism is equal to the altitude.*

(d) *The lateral faces of a prism are parallelograms.*

(e) *The lateral faces of a right prism are rectangles.*

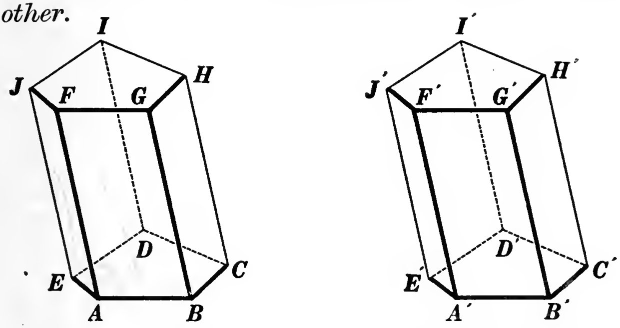
(f) *The bases of a prism are equal polygons.*

(g) *The sections of a prism made by two parallel planes cutting all the lateral edges are equal polygons.*

(h) *Every section of a prism made by a plane parallel to the base is equal to the base.*

PROPOSITION I. THEOREM

735. *Two prisms are equal if three faces including a trihedral angle of one are equal respectively, and similarly placed, to three faces including a trihedral angle of the other.*

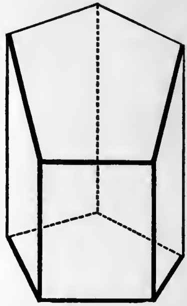


Given prisms AI and $A'I'$, face $AJ =$ face $A'J'$, face $AG =$ face $A'G'$, face $AD =$ face $A'D'$.

To prove prism $AI =$ prism $A'I'$.

ARGUMENT	REASONS
1. $\angle$s BAF, FAE, and BAE are equal, respectively, to $\angle$s $B'A'F'$, $F'A'E'$, and $B'A'E'$.	1. § 110.
2. $\therefore$ trihedral $\angle A =$ trihedral $\angle A'$.	2. § 704.
3. Place prism AI upon prism $A'I'$ so that trihedral $\angle A$ shall be superposed upon its equal, trihedral $\angle A'$.	3. § 54, 14.
4. Faces AJ, AG, and AD are equal, respectively, to faces $A'J'$, $A'G'$, and $A'D'$.	4. By hyp.
5. $\therefore$ J, F, and G will fall upon J', F', and G', respectively.	5. § 18.
6. CH and $C'H'$ are both $\parallel BG$.	6. § 734, *a*.
7. $\therefore$ CH and $C'H'$ are collinear.	7. § 179.
8. $\therefore$ H will fall upon H'.	8. § 603, *b*.
9. Likewise I will fall upon I'.	9. By steps similar to 6–8.
10. $\therefore$ prism $AI =$ prism $A'I'$. Q.E.D.	10. § 18.

736. Def. A **truncated prism** is the portion of a prism included between the base and a section of the prism made by a plane oblique to the base, but which cuts all the edges of the prism.



737. Cor. I. *Two truncated prisms are equal if three faces including a trihedral angle of one are equal respectively to three faces including a trihedral angle of the other, and the faces are similarly placed.*

738. Cor. II. *Two right prisms are equal if they have equal bases and equal altitudes.*

Ex. 1257. Two triangular prisms are equal if their lateral faces are equal, each to each.

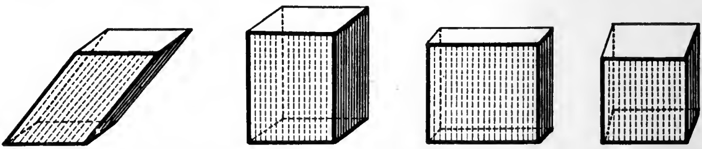
Ex. 1258. Classify the polyhedrons whose faces are: (*a*) four triangles; (*b*) two triangles and three parallelograms; (*c*) two quadrilaterals and four parallelograms; (*d*) two quadrilaterals and four rectangles; (*e*) two squares and four rectangles.

Ex. 1259. Find the sum of the plane angles of the dihedral angles whose edges are the lateral edges of a triangular prism; a quadrangular prism. (Hint. Draw a rt. section of the prism.)

Ex. 1260. Every section of a prism made by a plane parallel to a lateral edge is a parallelogram.

Ex. 1261. Every section of a prism made by a plane parallel to a lateral face is a parallelogram.

Ex. 1262. The section of a parallelopiped made by a plane passing through two diagonally opposite edges is a parallelogram.



Oblique Parallelopiped Right Parallelopiped Rectangular Parallelopiped Cube

739. Def. A **parallelopiped** is a prism whose bases are parallelograms.

740. Def. A **right parallelopiped** is a parallelopiped whose lateral edges are perpendicular to the bases.

741. Def. A **rectangular parallelopiped** is a right parallelopiped whose bases are rectangles.

742. Def. A **cube** (*i.e.* a regular hexahedron) is a rectangular parallelopiped whose edges are all equal.

743. The following are some of the properties of a parallelopiped; the student should prove the correctness of each:

(*a*) *All the faces of a parallelopiped are parallelograms.*

(*b*) *All the faces of a rectangular parallelopiped are rectangles.*

(*c*) *All the faces of a cube are squares.*

(*d*) *Any two opposite faces of a parallelopiped are equal and parallel.*

(*e*) *Any two opposite faces of a parallelopiped may be taken as the bases.*

Ex. 1263. Classify the polyhedrons whose faces are: (*a*) six parallelograms; (*b*) six rectangles; (*c*) six squares; (*d*) two parallelograms and four rectangles; (*e*) two rectangles and four parallelograms; (*f*) two squares and four rectangles.

Ex. 1264. Find the sum of all the face angles of a parallelopiped.

Ex. 1265. Find the diagonal of a cube whose edge is 8; 12; e.

Ex. 1266. Find the diagonal of a rectangular parallelopiped whose edges are 6, 8, and 12; whose edges are a, b, and c.

Ex. 1267. The edge of a cube : the diagonal of a face : the diagonal of the cube $= 1 : x : y$; find x and y.

Ex. 1268. Find the edge of a cube whose diagonal is $20\sqrt{3}$; d.

Ex. 1269. The diagonals of a rectangular parallelopiped are equal.

Ex. 1270. The diagonals of a parallelopiped bisect each other.

Ex. 1271. The diagonals of a parallelopiped meet in a point.

This point is sometimes called the **center** of the parallelopiped.

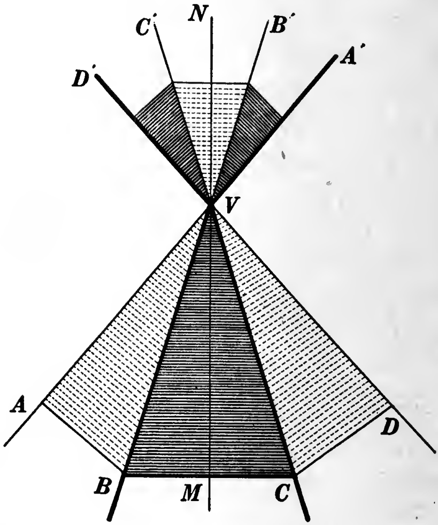
Ex. 1272. Any straight line through the center of a parallelopiped, with its extremities in the surface, is bisected at the center.

Ex. 1273. The sum of the squares of the four diagonals of a rectangular parallelopiped is equal to the sum of the squares of the twelve edges.

Ex. 1274. Is the statement in Ex. 1273 true for any parallelopiped?

PYRAMIDS

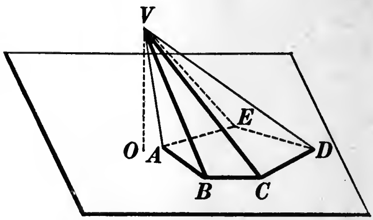
744. Def. A **pyramidal surface** is a surface generated by a moving straight line that continually intersects a fixed broken line and that passes through a fixed point not in the plane of the broken line.



745. Defs. By referring to §§ 693 and 694, give the definitions of **generatrix, directrix, vertex,** and **element** of a pyramidal surface. Point these out in the figure.

746. Def. A pyramidal surface consists of two parts lying on opposite sides of the vertex, called the **upper and lower nappes.**

747. Def. A **pyramid** is a polyhedron whose boundary consists of the portion of a pyramidal surface extending from its vertex to a plane cutting all its elements, and the section formed by this plane.



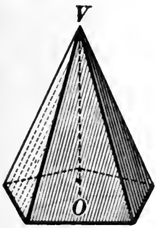
748. Defs. By referring to § 727, the student may give the definitions of **base, lateral faces,** and **lateral edges** of a pyramid. The vertex of the pyramidal surface is called the **vertex** of the pyramid, as *V*. Point these out in the figure.

In this text only pyramids whose bases are convex polygons will be considered.

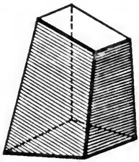
749. Defs. A pyramid is **triangular, quadrangular,** etc., according as its base is a triangle, a quadrilateral, etc.

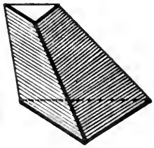
750. Questions. How many faces has a triangular pyramid? a tetrahedron? Can these terms be used interchangeably? How many different bases may a triangular pyramid have?

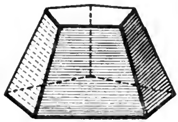
751. Def. The **altitude** of a pyramid is the perpendicular from the vertex to the plane of the base, as VO in the figure below, and in the figure on preceding page.



Regular Pyramid


Truncated Pyramid


Frustum of Triangular Pyramid


Frustum of Regular Pyramid

752. Def. A **regular pyramid** is a pyramid whose base is a regular polygon, and whose vertex lies in the perpendicular erected to the base at its center.

753. Def. A **truncated pyramid** is the portion of a pyramid included between the base and a section of the pyramid made by a plane cutting all the edges.

754. Def. A **frustum** of a pyramid is the portion of a pyramid included between the base and a section of the pyramid made by a plane parallel to the base.

755. The following are some of the properties of a pyramid; the student should prove the correctness of each:

(*a*) *The lateral edges of a regular pyramid are equal.*

(*b*) *The lateral edges of a frustum of a regular pyramid are equal.*

(*c*) *The lateral faces of a regular pyramid are equal isosceles triangles.*

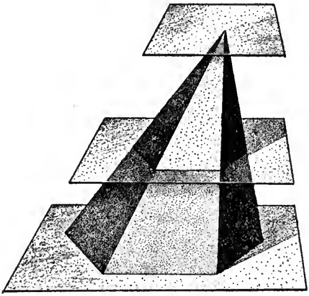
(*d*) *The lateral faces of a frustum of a regular pyramid are equal isosceles trapezoids.*

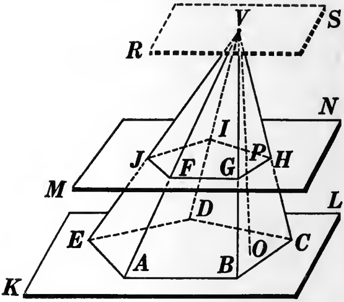
Proposition II. Theorem

756. *If a pyramid is cut by a plane parallel to the base:*

I. *The edges and altitude are divided proportionally.*

II. *The section is a polygon similar to the base.*





Given pyramid $V-ABCDE$ and plane $MN \parallel$ base AD cutting the lateral edges in F, G, H, I, and J and the altitude in P.

To prove: I. $\dfrac{VA}{VF}=\dfrac{VB}{VG}=\dfrac{VC}{VH}= \cdots =\dfrac{VO}{VP}$.

II. $FGHIJ \sim ABCDE$.

I. Argument	Reasons
1. Through V pass plane $RS \parallel$ plane KL.	1. § 652.
2. Then plane $RS \parallel$ plane MN.	2. § 654.
3. $\therefore \dfrac{VA}{VF}=\dfrac{VB}{VG}, \dfrac{VB}{VG}=\dfrac{VC}{VH}, \dfrac{VC}{VH}=\dfrac{VO}{VP}$, etc.	3. § 650.
4. $\therefore \dfrac{VA}{VF}=\dfrac{VB}{VG}=\dfrac{VC}{VH}= \cdots =\dfrac{VO}{VP}$. Q.E.D.	4. § 54, 1.

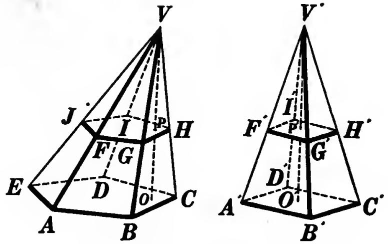
II. The proof of II is left as an exercise for the student.

757. Cor I. *Any section of a pyramid parallel to the base is to the base as the square of its distance from the vertex is to the square of the altitude of the pyramid.*

Hint. Prove $\dfrac{\overline{FG}^2}{\overline{AB}^2}=\dfrac{\overline{VG}^2}{\overline{VB}^2}=\dfrac{\overline{VP}^2}{\overline{VO}^2}$.

758. Cor. II. *If two pyramids having equal altitudes are cut by planes parallel to their bases, and at equal distances from their vertices, the sections have the same ratio as the bases.*

Hint. Apply § 757 to each pyramid.



759. Cor. III. *If two pyramids have equivalent bases and equal altitudes, sections made by planes parallel to the bases, and at equal distances from the vertices, are equivalent.*

Ex. 1275. Is every truncated pyramid a frustum of a pyramid? Is every frustum of a pyramid a truncated pyramid? What is the lower base of a frustum of a pyramid? the upper base? the altitude?

Ex. 1276. Classify the figures whose faces are as indicated below:
(*a*) one quadrilateral and four triangles;
(*b*) one square and four equal isosceles triangles;
(*c*) one pentagon and five triangles;
(*d*) two pentagons and five trapezoids;
(*e*) two squares and four equal isosceles trapezoids;
(*f*) two regular hexagons and six rectangles.

Ex. 1277. In the figure of § 758, if $VP = 12$, $PO = 8$, $VA = 28$, and $VB = 25$, find VF and VG.

Ex. 1278. The base of a pyramid, whose altitude is 2 decimeters, contains 200 square centimeters. Find the area of a section 6 centimeters from the vertex; 10 centimeters from the vertex.

Ex. 1279. The altitude of a pyramid with square base is 16 inches; the area of a section parallel to the base and 10 inches from the vertex is $56\frac{1}{4}$ square inches. Find the area of the base.

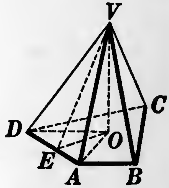
Ex. 1280. The altitude of a pyramid is H. At what distance from the vertex must a plane be passed parallel to the base so that the section formed shall be: (*a*) one half as large as the base? (*b*) one third? (*c*) one nth?

Ex. 1281. Prove that parallel sections of a pyramid are to each other as the squares of their distances from the vertex of the pyramid. Do the results obtained in Ex. 1280 fulfill this condition?

Ex. 1282. Each side of the base of a regular hexagonal pyramid is 6; the altitude is 15. How far from the vertex must a plane be passed parallel to the base to form a section whose area is $12\sqrt{3}$?

Ex. 1283. The areas of the bases of a frustum of a pyramid are 288 square feet and 450 square feet; the altitude of the frustum is 3 feet. Find the altitude of the pyramid of which the given figure is a frustum.

Ex. 1284. The bases of a frustum of a regular pyramid are equilateral triangles whose sides are 10 inches and 18 inches, respectively; the altitude of the frustum is 8 inches. Find the altitude of the pyramid of which the given figure is a frustum.



Ex. 1285. The sum of the lateral faces of any pyramid is greater than the base.

HINT. In the figure, let VE be the altitude of face VAD and VO the altitude of the pyramid. Which is the greater, VE or OE?

MENSURATION OF THE PRISM AND PYRAMID

AREAS

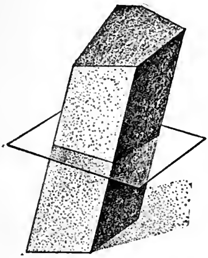
760. Def. The **lateral area** of a prism, a pyramid, or a frustum of a pyramid is the sum of the areas of its lateral faces.

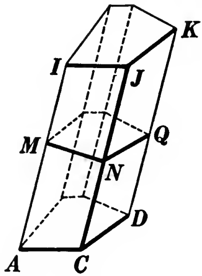
761. In the mensuration of the prism and pyramid the following notation will be used:

a, b, c = dimensions of a rectangular parallelopiped.
B = area of base in general or of lower base of a frustum.
b = area of upper base of a frustum.
E = lateral edge, or element, or edge of a tetrahedron in general.
H = altitude of a solid.
h = altitude of a surface.
L = slant height.
O = vertex of a pyramid.
P = perimeter of right section or of the lower base of a frustum.
p = perimeter of upper base of a frustum.
S = lateral area.
T = total area.
V = volume in general.
$v_1, v_2 \ldots$ = volumes of smaller solids into which a larger solid is divided.

Proposition III. Theorem

762. *The lateral area of a prism is equal to the product of the perimeter of a right section and a lateral edge.*





Given prism AK with MQ a rt. section, E a lateral edge, S the lateral area, and P the perimeter of rt. section MQ.

To prove $S = P \cdot E$.

Argument	Reasons
1. Rt. section $MQ \perp AI$, CJ, etc.	1. § 728.
2. $\therefore MN \perp AI$; $NQ \perp CJ$; etc.	2. § 619.
3. $\therefore MN$ is the altitude of $\square AJ$; NQ is the altitude of $\square CK$; etc.	3. § 228.
4. $\therefore$ area of $\square AJ = MN \cdot AI = MN \cdot E$; area of $\square CK = NQ \cdot CJ = NQ \cdot E$; etc.	4. § 481.
5. $\square AJ + \square CK + \cdots = (MN + NQ + \cdots)E$.	5. § 54, 2.
6. $\therefore S = P \cdot E$.	6. § 309.

763. Cor. *The lateral area of a right prism is equal to the product of the perimeter of its base and its altitude.*

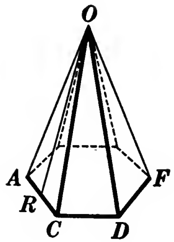
Hint. Thus, if $P =$ perimeter of base and $H =$ altitude, $S = P \cdot H$.

764. Def. The **slant height of a regular pyramid** is the altitude of any one of its triangular faces.

765. Def. The **slant height of a frustum** of a regular pyramid is the altitude of any one of its trapezoidal faces.

Proposition IV. Theorem

766. *The lateral area of a regular pyramid is equal to one half the product of the perimeter of its base and its slant height.*



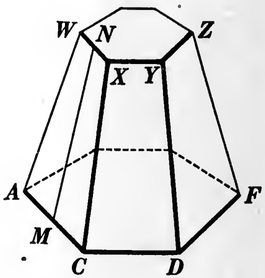
Given regular pyramid $O\text{–}ACD \cdots$ with the perimeter of its base denoted by P, its slant height by L, and its lateral area by S.

To prove $S = \frac{1}{2} P \cdot L$.

Argument	Reasons
1. Area of $\triangle AOC = \frac{1}{2} AC \cdot OR = \frac{1}{2} AC \cdot L$; area of $\triangle COD = \frac{1}{2} CD \cdot L$; etc.	1. § 485.
2. $\therefore \triangle AOC + \triangle COD + \cdots = \frac{1}{2}(AC + CD + \cdots)L$.	2. § 54, 2.
3. $\therefore S = \frac{1}{2} P \cdot L$. Q.E.D.	3. § 309.

767. Cor. *The lateral area of a frustum of a regular pyramid is equal to one half the product of the sum of the perimeters of its bases and its slant height.*

Hint. Prove $S = \frac{1}{2}(P + p)L$.



Ex. 1286. Find the lateral area and the total area of a regular pyramid each side of whose square base is 24 inches, and whose altitude is 16 inches.

Ex. 1287. The sides of the bases of a frustum of a regular octagonal pyramid are 15 centimeters and 24 centimeters, respectively, and the slant height is 30 centimeters. Find the number of square decimeters in the lateral area of the frustum.

Ex. 1288. Find the lateral area of a prism whose right section is a quadrilateral with sides 5, 7, 9, and 13 inches, and whose lateral edge is 15 inches.

Ex. 1289. Find the lateral area of a right prism whose altitude is 16 inches and whose base is a triangle with sides 8, 11, and 13 inches.

Ex. 1290. The perimeter of a right section of a prism is 45 decimeters; its altitude is $10\sqrt{3}$ decimeters; and a lateral edge makes with the base an angle of 60°. Find the lateral area.

Ex. 1291. Find the altitude of a regular prism, one side of whose triangular base is 5 inches and whose lateral area is 195 square inches.

Ex. 1292. Find the total area of a regular hexagonal prism whose altitude is 20 inches and one side of whose base is 10 inches.

Ex. 1293. Find the total area of a cube whose diagonal is $8\sqrt{3}$.

Ex. 1294. Find the edge of a cube if its total area is 294 square centimeters; if its total area is T.

Ex. 1295. Find the total area of a regular tetrahedron whose edge is 6 inches.

Ex. 1296. Find the lateral area and total area of a regular tetrahedron whose slant height is 8 inches.

Ex. 1297. Find the lateral area and total area of a regular hexagonal pyramid, a side of whose base is 6 inches and whose altitude is 10 inches.

Ex. 1298. Find the total area of a rectangular parallelopiped whose edges are 6, 8, and 12; whose edges are a, b, and c.

Ex. 1299. Find the total area of a right parallelopiped, one side of whose square base is 4 inches, and whose altitude is 6 inches.

Ex. 1300. The balcony of a theater is supported by four columns whose bases are regular hexagons. Find the cost, at 2 cents a square foot, of painting the columns if they are 20 feet high and the apothems of the bases are 10 inches.

Ex. 1301. In a frustum of a regular triangular pyramid, the sides of the bases are 8 and 4 inches, respectively, and the altitude is 10 inches. Find the slant height and a lateral edge.

Ex. 1302. In a frustum of a regular hexangular pyramid, the sides of the bases are 12 and 8, respectively, and the altitude is 16. Find the lateral area.

Ex. 1303. In a regular triangular pyramid, the side of the base is 8 inches, and the altitude is 12 inches; a lateral face makes with the base an angle of 60°. Find the lateral area.

Volumes

768. Note. The student should compare carefully §§ 769–776 with the corresponding discussion of the rectangle, §§ 466–473.

769. A solid may be **measured** by finding how many times it contains a *solid unit.* The solid unit most frequently chosen is a cube whose edge is of unit length. If the unit length is an inch, the solid unit is a cube whose edge is an inch. Such a unit is called a **cubic inch.** If the unit length is a foot, the solid unit is a cube whose edge is a foot, and the unit is called a **cubic foot.**

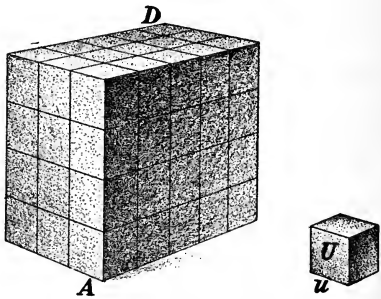


FIG. 1. Rectangular Parallelopiped $AD = 60\ U$.

770. Def. The result of the measurement is a *number,* which is called the **measure-number,** or **numerical measure,** or **volume** of the solid.

771. Thus, if the unit cube U is contained in the rectangular

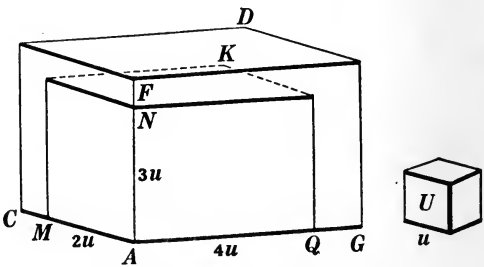


FIG. 2. Rectangular Parallelopiped $AD = 24\ U+$.

parallelopiped *AD* (Fig. 1) 60 times, then the measure-number or volume of rectangular parallelopiped *AD*, in terms of *U*, is 60.

If the given unit cube is not contained in the given rectangular parallelopiped an integral number of times without a remainder (Fig. 2), then by taking a cube that is an aliquot part of *U*, as one eighth of *U*, and applying it as a measure to the rectangular parallelopiped (Fig. 3), a number will be obtained

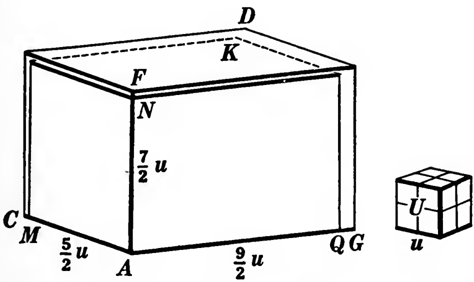


FIG. 3. Rectangular Parallelopiped $AD = \frac{315}{8}\,U+ = 39\frac{3}{8}\,U+$.

which, divided by 8,* will give another (and usually closer) approximate volume of the given rectangular parallelopiped. By proceeding in this way (Fig. 4), closer and closer approximations to the true volume may be obtained.

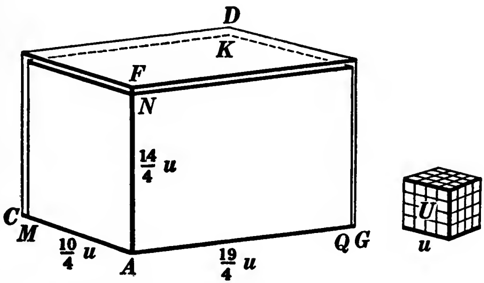


FIG. 4. Rectangular Parallelopiped $AD = \frac{2660}{64}\,U+ = 41\frac{9}{16}\,U+$.

* It takes eight of the small cubes to make the *unit cube* itself.

772. If the edges of the given rectangular parallelopiped and the edge of the unit cube are *commensurable,* a cube may be found which is an aliquot part of U, and which will be contained in the rectangular parallelopiped an integral number of times.

773. If the edges of the given rectangular parallelopiped and the edge of the unit cube are *incommensurable,* then closer and closer approximations to the volume may be obtained, but no cube which is an aliquot part of U will be also an aliquot part of the rectangular parallelopiped (by definition of incommensurable magnitudes).

There is, however, a definite *limit* which is approached more and more closely by the approximations obtained by using smaller and smaller subdivisions of the unit cube, as these subdivisions approach zero as a limit.

774. Def. The **volume** of a rectangular parallelopiped which is incommensurable with the chosen unit cube is the *limit* which successive approximate volumes of the rectangular parallelopiped *approach* as the subdivisions of the unit cube approach zero as a limit.

For brevity the expression *the volume of a solid,* or simply *the solid,* is used to mean the volume of the solid with respect to a chosen unit.

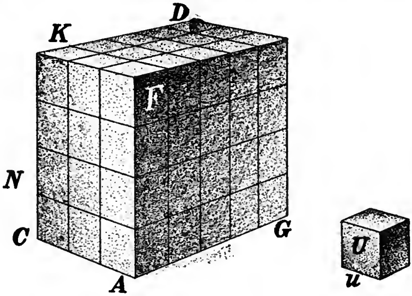
775. Def. The **ratio** of any two solids is the ratio of their measure-numbers, or volumes (based on the same unit).

776. Def. Two solids are **equivalent** if their volumes are equal.

777. Historical Note. The determination of the volumes of polyhedrons is found in a document as ancient as the Rhind papyrus, which is thought to be a copy of a manuscript dating back possibly as far as 3400 B.C. (See § 474.) In this manuscript Ahmes calculates the contents of an Egyptian barn by means of the formula, $V = a \cdot b \cdot (c + \frac{1}{2} c)$, where a, b, and c are supposed to be linear dimensions of the barn. But unfortunately the exact shape of these barns is unknown, so that the accuracy of the formula cannot be tested.

Proposition V. Theorem

778. *The volume of a rectangular parallelopiped is equal to the product of its three dimensions.*



Given rectangular parallelopiped AD, with dimensions AC, AF, and AG; and U the chosen unit of volume, whose edge is u.

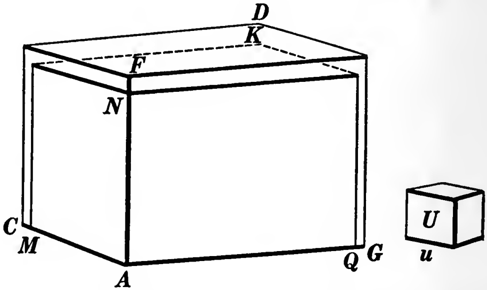
To prove the volume of $AD = AC \cdot AF \cdot AG$.

I. If AC, AF, and AG are each commensurable with u.

(*a*) Suppose that u is contained in AC, AF, and AG each an integral number of times.

Argument	Reasons
1. Lay off u upon AC, AF, and AG. Suppose that u is contained in AC r times, in AF s times, and in AG t times.	1. § 335.
2. At the points of division on AC, AF, and AG draw planes $\perp$ AC, AF, and AG.	2. § 627.
3. Then AD is divided into unit cubes.	3. § 769.
4. There are r of these unit cubes in a row along AC, s of these rows in parallelopiped AK, and t such parallelopipeds in parallelopiped AD.	4. Arg. 1.
5. $\therefore$ the volume of $AD = r \cdot s \cdot t$.	5. § 770.
6. But r, s, and t are the measure-numbers of AC, AF, and AG, respectively, referred to the linear unit u.	6. Arg. 1.
7. $\therefore$ the volume of $AD = AC \cdot AF \cdot AG$. Q.E.D.	7. § 309.

(*b*) Suppose that u is not a measure of AC, AF, and AG, respectively, but that some aliquot part of u is such a measure. The proof is left as an exercise for the student.



II. If AC, AF, and AG are each incommensurable with u.

Argument	Reasons
1. Let m be a measure of u. Apply m as a measure to AC, AF, and AG, respectively, as many times as possible. There will be remainders, as MC, NF, and QG, each less than m.	1. § 339.
2. Through M draw plane $MK \perp AC$, through N draw plane $NK \perp AF$, and through Q draw plane $QK \perp AG$.	2. § 627.
3. Now AM, AN, and AQ are each commensurable with the measure m, and hence with u, the linear unit.	3. § 337.
4. $\therefore$ the volume of rectangular parallelopiped $AK = AM \cdot AN \cdot AQ$.	4. § 778, I.
5. Now take a smaller measure of u. No matter how small a measure of u is taken, when it is applied as a measure to AC, AF, and AG, the remainders, MC, NF, and QG, will be smaller than the measure taken.	5. § 335.

Argument	Reasons
6. ∴ the difference between AM and AC, the difference between AN and AF, and the difference between AQ and AG, may each be made to become and remain less than any previously assigned segment, however small.	6. Arg. 5.
7. ∴ AM approaches AC as a limit, AN approaches AF as a limit, and AQ approaches AG as a limit.	7. § 349.
8. ∴ $AM \cdot AN \cdot AQ$ approaches $AC \cdot AF \cdot AG$ as a limit.	8. § 593.
9. Again, the difference between rectangular parallelopiped AK and rectangular parallelopiped AD may be made to become and remain less than any previously assigned volume, however small.	9. Arg. 5.
10. ∴ the volume of rectangular parallelopiped AK approaches the volume of rectangular parallelopiped AD as a limit.	10. § 349.
11. But the volume of AK is always equal to $AM \cdot AN \cdot AQ$.	11. Arg. 4.
12. ∴ the volume of $AD = AC \cdot AF \cdot AG$. Q.E.D.	12. § 355.

III. If AC is commensurable with u but AF and AG are incommensurable with u.

IV. If AC and AF are commensurable with u but AG is incommensurable with u.

The proofs of III and IV are left as exercises for the student.

779. Cor I. *The volume of a cube is equal to the cube of its edge.*

Hint. Compare with § 478.

780. Cor. II. *Any two rectangular parallelopipeds are to each other as the products of their three dimensions.* (HINT. Compare with § 479.)

781. Note. By the product of a surface and a line is meant the product of the measure-numbers of the surface and the line.

782. Cor. III. *The volume of a rectangular parallelopiped is equal to the product of its base and its altitude.*

783. Cor. IV. *Any two rectangular parallelopipeds are to each other as the products of their bases and their altitudes.* (HINT. Compare with § 479.)

784. Cor. V. (*a*) *Two rectangular parallelopipeds having equivalent bases are to each other as their altitudes;* (*b*) *two rectangular parallelopipeds having equal altitudes are to each other as their bases.* (HINT. Compare with § 480.)

785. Cor. VI. (*a*) *Two rectangular parallelopipeds having two dimensions in common are to each other as their third dimensions, and* (*b*) *two rectangular parallelopipeds having one dimension in common are to each other as the products of their other two dimensions.*

786. Questions. What is it in Book IV that corresponds to volume in Book VII? to rectangular parallelopiped? State the theorem and corollaries in Book IV that correspond to §§ 778, 779, 782, 783, and 784. Will the proofs given there, with the corresponding changes in terms, apply here? Compare the entire discussion of §§ 466–480 with §§ 769–785.

Ex. 1304. Find the volume of a cube whose diagonal is $5\sqrt{3}$; d.

Ex. 1305. The volume of a rectangular parallelopiped is V; each side of the square base is one third the altitude of the parallelopiped. Find the side of the base. Find the side of the base if $V = 192$ cubic feet.

Ex. 1306. The dimensions of two rectangular parallelopipeds are 6, 8, 10 and 5, 12, 16, respectively. Find the ratio of their volumes.

Ex. 1307. The total area of a cube is 300 square inches; find its volume.

Ex. 1308. The volume of a certain cube is V; find the volume of a cube whose edge is twice that of the given cube.

Ex. 1309. The edge of a cube is a; find the edge of a cube twice as large; *i.e.* containing twice the volume of the given cube.

787. Historical Note. Plato (429–348 B.C.) was one of the first to discover a solution to that famous problem of antiquity, the *duplication of a cube, i.e.* the finding of the edge of a cube whose volume is double that of a given cube.

PLATO

There are two legends as to the origin of the problem. The one is that an old tragic poet represented King Minos as wishing to erect a tomb for his son Glaucus. The king being dissatisfied with the dimensions (100 feet each way) proposed by his architect, exclaimed: "The inclosure is too small for a royal tomb; double it, but fail not in the cubical form."

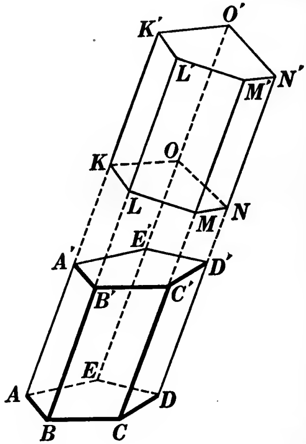
The other legend asserts that the Athenians, who were suffering from a plague of typhoid fever, consulted the oracle at Delos as to how to stop the plague. Apollo replied that the Delians would have to double the size of his altar, which was in the form of a cube. A new altar was constructed having its edge twice as long as that of the old one. The pestilence became worse than before, whereupon the Delians appealed to Plato. It is therefore known as the Delian problem.

Plato was born in Athens, and for eight years was a pupil of Socrates. Plato possessed considerable wealth, and after the death of Socrates in 399 B.C. he spent some years in traveling and in the study of mathematics. It was during this time that he became acquainted with the members of the Pythagorean School, especially with Archytas, who was then its head. No doubt it was his association with these people that gave him his passion for mathematics. About 380 B.C. he returned to his native city, where he established a school. Over the entrance to his school was this inscription: "Let none ignorant of geometry enter my door." Later an applicant who knew no geometry was actually turned away with the statement: "Depart, for thou hast not the grip of philosophy."

Plato is noted as a teacher, rather than an original discoverer, and his contributions to geometry are improvements in its method rather than additions to its matter. He valued geometry mainly as a "means of education in right seeing and thinking and in the conception of imaginary processes." It is stated on good authority that "Plato was almost as important as Pythagoras to the advance of Greek geometry."

Proposition VI. Theorem

788. *An oblique prism is equivalent to a right prism, whose base is a right section of the oblique prism, and whose altitude is equal to a lateral edge of the oblique prism.*



Given oblique prism AD'; also rt. prism KN' with base KN a rt. section of AD', and with KK', LL', etc., lateral edges of KN', equal to AA', BB', etc., lateral edges of AD'.

To prove oblique prism $AD' \approx$ rt. prism KN'.

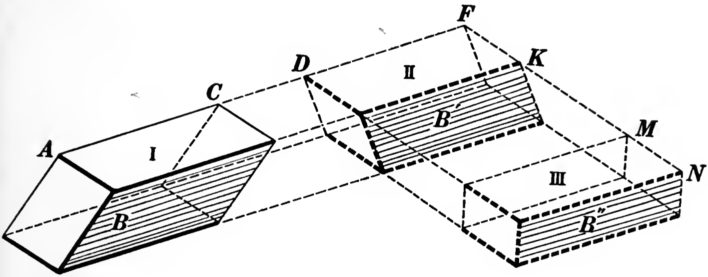
Outline of Proof

1. In truncated prisms AN and $A'N'$, prove the $\angle$s of face BK equal, respectively, to the $\angle$s of face $B'K'$.
2. Prove the sides of face BK equal, respectively, to the sides of face $B'K'$.
3. $\therefore$ face BK = face $B'K'$.
4. Similarly face BM = face $B'M'$, and face BE = face $B'E'$.
5. $\therefore$ truncated prism AN = truncated prism $A'N'$ (§ 737).
6. But truncated prism $A'N$ = truncated prism $A'N$.
7. $\therefore$ oblique prism $AD' \approx$ rt. prism KN'. Q.E.D.

789. Question. Is there a theorem in Book IV that corresponds to Prop. VI? If not, formulate one and see if you can prove it true.

Proposition VII. Theorem

790. *The volume of any parallelopiped is equal to the product of its base and its altitude.*



Given parallelopiped I with its volume denoted by V, its base by B, and its altitude by H.

To prove $V = B \cdot H$.

Argument	Reasons
1. Prolong edge AC and all edges of $I \parallel AC$.	1. § 54, 16.
2. On the prolongation of AC take $DF = AC$, and through D and F pass planes $\perp AF$, forming rt. parallelopiped II.	2. § 627.
3. Then $I \approx II$.	3. § 788.
4. Prolong edge FK and all edges of $II \parallel FK$.	4. § 54, 16.
5. On the prolongation of FK take $MN = FK$, and through M and N pass planes $\perp FN$, forming rectangular parallelopiped III.	5. § 627.
6. Then $II \approx III$.	6. § 788.
7. $\therefore I \approx III$.	7. § 54, 1.
8. Again, $B \approx B' = B''$.	8. § 482.
9. Also H, the altitude of I, = the altitude of III.	9. § 663.
10. But the volume of $III = B'' \cdot H$.	10. § 782.
11. $\therefore V = B \cdot H$. Q.E.D.	11. § 309.

791. Cor. I. *Parallelopipeds having equivalent bases and equal altitudes are equivalent.*

792. Cor. II. *Any two parallelopipeds are to each other as the products of their bases and their altitudes.*

793. Cor. III. (*a*) *Two parallelopipeds having equivalent bases are to each other as their altitudes, and* (*b*) *two parallelopipeds having equal altitudes are to each other as their bases.*

794. Questions. What expression in Book IV corresponds to *volume of a parallelopiped ?* Quote the theorem and corollaries in Book IV that correspond to §§ 790–793. Will the proofs given there, with the corresponding changes, apply here ?

Ex. 1310. Prove Prop. VI by subtracting the equal truncated prisms of Arg. 5 from the entire figure.

Ex. 1311. The base of a parallelopiped is a parallelogram two adjacent sides of which are 8 and 15, respectively, and they include an angle of 30°. If the altitude of the parallelopiped is 10, find its volume.

Ex. 1312. Four parallelopipeds have equivalent bases and equal lateral edges. In the first the lateral edge makes with the base an angle of 30° ; in the second an angle of 45° ; in the third an angle of 60° ; and in the fourth an angle of 90°. Find the ratio of the volumes of the four parallelopipeds.

Ex. 1313. Find the edge of a cube equivalent to a rectangular parallelopiped whose edges are 6, 10, and 15 ; whose edges are a, b, and c.

Ex. 1314. Find the diagonal of a cube whose volume is 512 cubic inches ; a cubic inches.

Ex. 1315. The edge of a cube is a. Find the area of a section made by a plane through two diagonally opposite edges.

Ex. 1316. How many cubic feet of cement will be needed to make a box, including lid, if the inside dimensions of the box are 2 feet 6 inches, 3 feet, and 4 feet 6 inches, if the cement is 3 inches thick ?

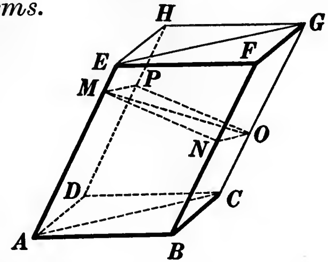
Hint. In a problem of this kind, always find the volume of the whole solid, and the volume of the inside solid, then subtract.

Ex. 1317. The volume of a rectangular parallelopiped is 2430 cubic inches, and its edges are in the ratio of 3, 5, and 6. Find its edges.

Ex. 1318. In a certain cube the area of the surface and the volume have the same numerical value. Find the volume of the cube.

PROPOSITION VIII. THEOREM

795. *The plane passed through two diagonally opposite edges of a parallelopiped divides it into two equivalent triangular prisms.*



Given plane AG passed through edges AE and CG of parallelopiped BH dividing it into the two triangular prisms $ABC\text{–}F$ and $CDA\text{–}H$.

To prove prism $ABC\text{–}F \approx$ prism $CDA\text{–}H$.

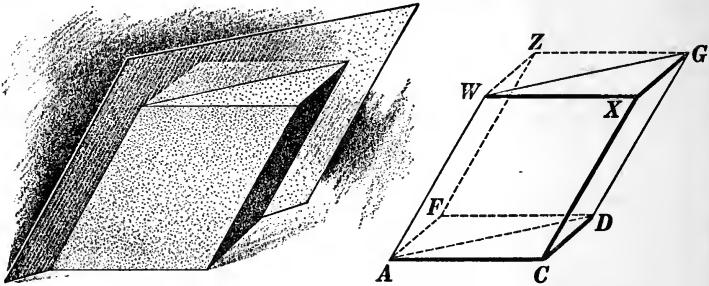
ARGUMENT ONLY

1. Let $MNOP$ be a rt. section of parallelopiped BH, cutting the plane AG in line MO.
2. Face $AF \parallel$ face DG and face $AH \parallel$ face BG.
3. $\therefore MN \parallel PO$ and $MP \parallel NO$.
4. $\therefore MNOP$ is a ▱.
5. $\therefore \triangle MNO = \triangle OPM$.
6. Now triangular prism $ABC\text{–}F \approx$ a rt. prism whose base is $\triangle MNO$, a rt. section of prism $ABC\text{–}F$, and whose altitude is AE, a lateral edge of prism $ABC\text{–}F$.
7. Likewise triangular prism $CDA\text{–}H \approx$ a rt. prism whose base is $\triangle OPM$ and whose altitude is AE.
8. But two such prisms are equivalent.
9. $\therefore$ prism $ABC\text{–}F \approx$ prism $CDA\text{–}H$. Q.E.D.

796. Questions. Is there a theorem in Book I that corresponds to Prop. VIII? If so, state it. Could an oblique prism exist such that a right section, as $MNOP$, might intersect either base? If so, draw a figure to illustrate.

Proposition IX. Theorem

797. *The volume of a triangular prism is equal to the product of its base and its altitude.*



Given triangular prism $ACD\text{–}X$ with its volume denoted by V, its base by B, and its altitude by H.

To prove $V = B \cdot H$.

The proof is left as an exercise for the student.

798. Questions. What proposition in Book IV corresponds to Prop. IX above? Can you apply the proof there given? What is the name of the figure CZ in § 797? What is its volume? What part of CZ is $ACD\text{–}X$ (§ 795)?

Ex. 1319. The volume of a triangular prism is equal to one half the product of any lateral face and the perpendicular from any point in the opposite edge to that face.

Hint. The triangular prism is one half of a certain parallelopiped (§ 795).

Ex. 1320. The base of a coal bin which is 8 feet deep is a triangle with sides 10 feet, 15 feet, and 20 feet, respectively. How many tons of coal will the bin hold considering 35 cubic feet of coal to a ton?

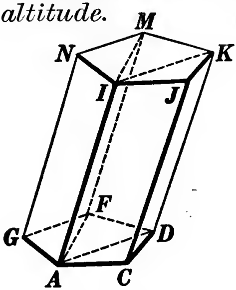
Ex. 1321. One face of a triangular prism contains 45 square inches; the perpendicular to this face from a point in the opposite edge is 6 inches. Find the volume of the prism.

Ex. 1322. During a rainfall of $\frac{1}{2}$ inch, how many barrels of water will fall upon a ten-acre field, counting $7\frac{1}{2}$ gallons to a cubic foot and $31\frac{1}{2}$ gallons to a barrel?

Ex. 1323. The inside dimensions of an open tank before lining are 6 feet, 2 feet 6 inches, and 2 feet, respectively, the latter being the height. Find the number of pounds of zinc required to line the tank with a coating $\frac{1}{4}$ inch thick, a cubic foot of zinc weighing 6860 ounces.

PROPOSITION X. THEOREM

799. *The volume of any prism is equal to the product of its base and its altitude.*



Given prism AM with its volume denoted by V, its base by B, and its altitude by H.

To prove $V = B \cdot H$.

ARGUMENT	REASONS
1. From any vertex of the lower base, as A, draw diagonals AD, AF, etc.	1. § 54, 15.
2. Through edge AI and these diagonals pass planes AK, AM, etc.	2. § 612.
3. Prism AM is thus divided into triangular prisms.	3. § 732.
4. Denote the volume and base of triangular prism $ACD–J$ by v_1 and b_1; of $ADF–K$ by v_2 and b_2; etc. Then $v_1 = b_1H$; $v_2 = b_2H$; etc.	4. § 797.
5. $\therefore v_1 + v_2 + \cdots = (b_1 + b_2 + \cdots)H$.	5. § 54, 2.
6. $\therefore$ $V = B \cdot H$. Q.E.D.	6. § 309.

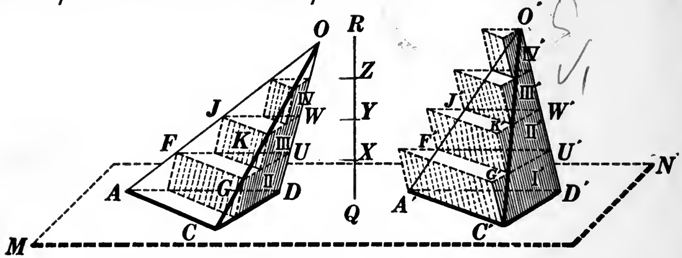
800. Cor. I. *Prisms having equivalent bases and equal altitudes are equivalent.*

801. Cor. II. *Any two prisms are to each other as the products of their bases and their altitudes.*

802. Cor. III. *(a) Two prisms having equivalent bases are to each other as their altitudes: (b) two prisms having equal altitudes are to each other as their bases.*

Proposition XI. Theorem

803. *Two triangular pyramids having equivalent bases and equal altitudes are equivalent.*



Given triangular pyramids $O\text{–}ACD$ and $O'\text{–}A'C'D'$ with base $ACD \approx$ base $A'C'D'$, with altitudes each equal to QR, and with volumes denoted by V and V', respectively.

To prove $V = V'$.

Argument	Reasons
1. $V = V'$, $V < V'$, or $V > V'$.	1. 161, *a*.
2. Suppose $V < V'$, so that $V' - V = k$, a constant. For convenience, place the two pyramids so that their bases are in the same plane, MN.	2. § 54, 14.
3. Divide the common altitude QR into n equal parts, as QX, XY, etc., and through the several points of division pass planes ∥ plane MN.	3. § 653.
4. Then section $FGU \approx$ section $F'G'U'$, section $JKW \approx$ section $J'K'W'$, etc.	4. § 759.
5. On FGU, JKW, etc., as upper bases, construct prisms with edges ∥ DO and with altitudes $= QX$. Denote these prisms by II, III, etc.	5. § 726.
6. On $A'C'D'$, $F'G'U'$, etc., as lower bases, construct prisms with edges ∥ $D'O'$ and with altitudes $= QX$. Denote these prisms by I′, II′, etc.	6. § 726.

Argument	Reasons
7. Then prism II $\approx$ prism II$'$, prism III $\approx$ prism III$'$, etc.	7. § 800.
8. Now denote the sum of the volumes of prisms II, III, etc., by S; the sum of the volumes of prisms I$'$, II$'$, III$'$, etc., by S'; and the volume of prism I$'$ by v'. Then $S' - S = v'$.	8. § 54, 3.
9. But $V' < S'$ and $S < V$.	9. § 54, 12.
10. $\therefore V' + S < V + S'$.	10. § 54, 9.
11. $\therefore V' - V < S' - S$; *i.e.* $V' - V < v'$.	11. § 54, 5.
12. By making the divisions of the altitude QR smaller and smaller, prism I$'$, and hence v' may be made less than any previously assigned volume, however small.	12. 802, *a*.
13. $\therefore V' - V$, which is $< v'$, may be made less than any previously assigned volume, however small.	13. § 54, 10.
14. $\therefore$ the supposition that $V' - V = k$, a constant, is false; *i.e.* V is not $< V'$.	14. Arg. 13.
15. Similarly it may be proved that V' is not $< V$.	15. By steps similar to 2–14.
16. $\therefore V = V'$. Q.E.D.	16. § 161, *b*.

Ex. 1324. The volume of an oblique prism is equal to the product of its right section and a lateral edge.

Hint. Apply § 788.

Ex. 1325. The volume of a regular prism is equal to the product of its lateral area and one half the apothem of its base.

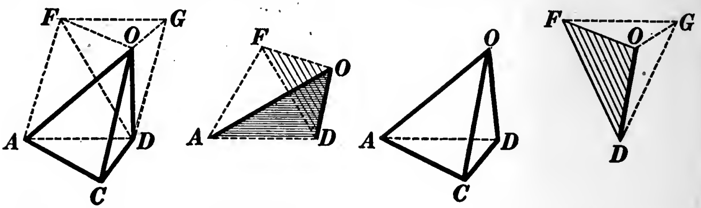
Hint. See Ex. 1319.

Ex. 1326. The base of a prism is a rhombus having one side 29 inches and one diagonal 42 inches. If the altitude of the prism is 25 inches, find its volume.

Ex. 1327. In a certain cube the area of the surface and the combined lengths of its edges have the same numerical value. Find the volume of the cube.

Proposition XII. Theorem

804. *The volume of a triangular pyramid is equal to one third the product of its base and its altitude.*



Given triangular pyramid $O\text{–}ACD$ with its volume denoted by V, its base by B, and its altitude by H.

To prove $V = \frac{1}{3} B \cdot H$.

Argument Only

1. Construct prism AG with base ACD and lateral edge CO.

2. The prism is then composed of triangular pyramid $O\text{–}ACD$ and quadrangular pyramid $O\text{–}ADGF$.

3. Through OD and OF pass a plane, intersecting $ADGF$ in DF and dividing quadrangular pyramid $O\text{–}ADGF$ into two triangular pyramids $O\text{–}ADF$ and $O\text{–}DGF$.

4. $ADGF$ is a $\square$; $\therefore \triangle ADF = \triangle DGF$.

5. $\therefore O\text{–}ADF \Bumpeq O\text{–}DGF$.

6. But in triangular pyramid $O\text{–}DGF$, OGF may be taken as base and D as vertex; then $O\text{–}DGF = D\text{–}OGF \Bumpeq O\text{–}ACD$.

7. But $O\text{–}ACD + O\text{–}ADF + O\text{–}DGF \Bumpeq$ prism AG.

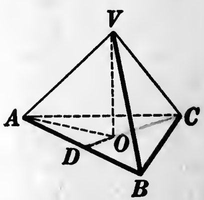
8. $\therefore$ 3 times the volume of $O\text{–}ACD =$ the volume of prism AG.

9. $\therefore V = \frac{1}{3}$ the volume of prism AG.

10. But prism $AG = B \cdot H$; $\therefore V = \frac{1}{3} B \cdot H$. Q.E.D.

Ex. 1328. Find the volume of a regular tetrahedron whose edge is 6.

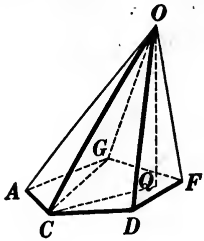
Hint. O, the foot of the $\perp$ from V to the plane of base ABC, is the center of $\triangle ABC$ (§ 752). Hence $OA = \frac{2}{3}$ of the altitude of $\triangle ABC$, and a $\perp$ from O to any edge of the base, as $OD = \frac{1}{2}$ of OA.



Ex. 1329. Find the volume of a regular tetrahedron with slant height $2\sqrt{3}$; with altitude a.

PROPOSITION XIII. THEOREM

805. *The volume of any pyramid is equal to one third the product of its base and its altitude.*



Given pyramid $O\text{–}ACDFG$ with its volume denoted by V, its base by B, and its altitude, OQ, by H.

To prove $V = \frac{1}{3} B \cdot H$.

The proof is left as an exercise for the student.

HINT. See proof of Prop. X.

806. Cor. I. *Pyramids having equivalent bases and equal altitudes are equivalent.*

807. Cor. II. *Any two pyramids are to each other as the products of their bases and their altitudes.*

808. Cor. III. (*a*) *Two pyramids having equivalent bases are to each other as their altitudes, and* (*b*) *two pyramids having equal altitudes are to each other as their bases.*

Ex. 1330. In the figure of § 805, if the base = 250 square inches, OC = 18 inches, and the inclination of OC to the base is 60°, find the volume.

Ex. 1331. A pyramid and a prism have equivalent bases and equal altitudes; find the ratio of their volumes.

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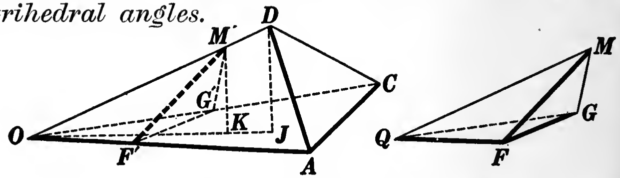
809. Historical Note. The proof of the proposition that "every pyramid is the third part of a prism on the same base and with the same altitude" is attributed to Eudoxus (408 – 355 B.C.), a great mathematician of the Athenian School. In a noted work written by Archimedes (287–212 B.C.), called *Sphere and Cylinder*, there is also found an expression for the surface and volume of a pyramid. (For a further account of Archimedes, see §§ 542, 896, and 973.) Later a solution of this problem was given by Brahmagupta, a noted Hindoo writer born about 598 A.D.

Proposition XIV. Theorem

810. *Two triangular pyramids, having a trihedral angle of one equal to a trihedral angle of the other, are to each other as the products of the edges including the equal trihedral angles.*



Given triangular pyramids $O\text{–}ACD$ and $Q\text{–}FGM$ with trihedral $\angle O =$ trihedral $\angle Q$, and with volumes denoted by V and V', respectively.

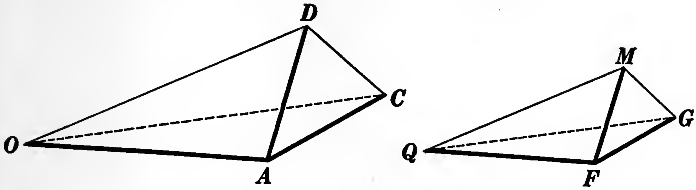
To prove $\dfrac{V}{V'} = \dfrac{OA \cdot OC \cdot OD}{QF \cdot QG \cdot QM}$.

Argument	Reasons
1. Place pyramid $Q\text{–}FGM$ so that trihedral $\angle Q$ shall coincide with trihedral $\angle O$. Represent pyramid $Q\text{–}FGM$ in its new position by $O\text{–}F'G'M'$.	1. § 54, 14.
2. From D and M' draw DJ and $M'K \perp$ plane OAC.	2. § 639.
3. Then $\dfrac{V}{V'} = \dfrac{\triangle OAC \cdot DJ}{\triangle OF'G' \cdot M'K} = \dfrac{\triangle OAC}{\triangle OF'G'} \cdot \dfrac{DJ}{M'K}$.	3. § 807.
4. But $\dfrac{\triangle OAC}{\triangle OF'G'} = \dfrac{OA \cdot OC}{OF' \cdot OG'}$.	4. § 498.
5. Again let the plane determined by DJ and $M'K$ intersect plane OAC in line OKJ.	5. §§ 613, 616.
6. Then rt. $\triangle DJO \sim$ rt. $\triangle M'KO$.	6. § 422.
7. $\therefore \dfrac{DJ}{M'K} = \dfrac{OD}{OM'}$.	7. § 424, 2.
8. $\therefore \dfrac{V}{V'} = \dfrac{OA \cdot OC}{OF' \cdot OG'} \cdot \dfrac{OD}{OM'} = \dfrac{OA \cdot OC \cdot OD}{QF \cdot QG \cdot QM}$. Q.E.D.	8. § 309.

811. Def. Two polyhedrons are **similar** if they have the same number of faces similar each to each and similarly placed, and have their corresponding polyhedral angles equal.

Proposition XV. Theorem

812. *The volumes of two similar tetrahedrons are to each other as the cubes of any two homologous edges.*



Given similar tetrahedrons $O\text{–}ACD$ and $Q\text{–}FGM$ with volumes denoted by V and V', and with OA and QF two homologous edges.

To prove $\dfrac{V}{V'} = \dfrac{\overline{OA}^3}{\overline{QF}^3}$.

Argument	Reasons
1. Trihedral $\angle O =$ trihedral $\angle Q$.	1. § 811.
2. $\therefore \dfrac{V}{V'} = \dfrac{OA \cdot OC \cdot OD}{QF \cdot QG \cdot QM} = \dfrac{OA}{QF} \cdot \dfrac{OC}{QG} \cdot \dfrac{OD}{QM}$.	2. § 810.
3. But $\dfrac{OA}{QF} = \dfrac{OC}{QG} = \dfrac{OD}{QM}$.	3. § 424, 2.
4. $\therefore \dfrac{V}{V'} = \dfrac{OA}{QF} \cdot \dfrac{OA}{QF} \cdot \dfrac{OA}{QF} = \dfrac{\overline{OA}^3}{\overline{QF}^3}$. Q.E.D.	4. § 309.

813. Question. Compare §§ 810 and 812 with §§ 498 and 503. Are the same general methods used in the two sets of theorems?

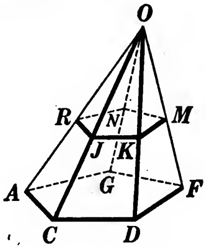
814. Note. The proposition, "two similar convex polyhedrons are to each other as the cubes of any two homologous edges," will be assumed at this point, and will be applied in some of the exercises that follow. For a complete discussion of this principle see Appendix, §§ 1022–1029.

Ex. 1332. The edges of two regular tetrahedrons are 6 centimeters and 8 centimeters, respectively. Find the ratio of their volumes.

Ex. 1333. The volumes of two similar polyhedrons are 343 cubic inches and 512 cubic inches, respectively: (*a*) an edge of the first figure is 14 inches, find the homologous edge of the second; (*b*) the total area of the first figure is 280 square inches, find the total area of the second.

PROPOSITION XVI. THEOREM

815. *The volume of a frustum of any pyramid is equal to one third the product of its altitude and the sum of its lower base, its upper base, and the mean proportional between its two bases.*



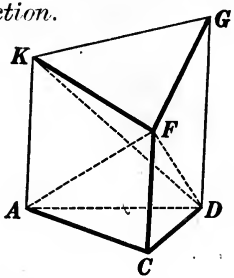
Given frustum AM, of pyramid $O\text{–}AF$, with its volume denoted by V, its lower base by B, its upper base by b, and its altitude by H.

To prove $V = \frac{1}{3} H(B + b + \sqrt{B \cdot b})$.

ARGUMENT	REASONS
1. Frustum AM = pyramid $O\text{–}AF$ minus pyramid $O\text{–}RM$.	1. § 54, 11.
2. Let H' denote the altitude of $O\text{–}RM$. Then $V = \frac{1}{3} B(H + H') - \frac{1}{3} b \cdot H'$ $= \frac{1}{3} HB + \frac{1}{3} H'(B - b)$. It now remains to find the value of H'.	2. § 805.
3. $\dfrac{b}{B} = \dfrac{H'^2}{(H + H')^2}$.	3. § 757.
4. $\therefore \dfrac{\sqrt{b}}{\sqrt{B}} = \dfrac{H'}{H + H'}$.	4. § 54, 13.
5. Whence $H' = \dfrac{H\sqrt{b}}{\sqrt{B} - \sqrt{b}}$.	5. Solving for H'.
6. $\therefore V = \frac{1}{3} HB + \frac{1}{3} \dfrac{H\sqrt{b}}{\sqrt{B} - \sqrt{b}}(B - b)$ $= \frac{1}{3} HB + \frac{1}{3} Hb + \frac{1}{3} H\sqrt{B \cdot b}$; *i.e.* $V = \frac{1}{3} H(B + b + \sqrt{B \cdot b})$. Q.E.D.	6. § 309.

Proposition XVII. Theorem

816. *A truncated triangular prism is equivalent to three triangular pyramids whose bases are the base of the frustum and whose vertices are the three vertices of the inclined section.*



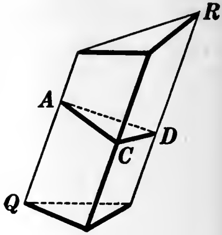
Given truncated triangular prism *ACD–FGK.*

To prove *ACD–FGK* ≎ *F–ACD* + *G–ACD* + *K–ACD.*

Argument	Reasons
1. Through *A, D, F* and *K, D, F* pass planes dividing frustum *ACD–FGK* into three triangular pyramids *F–ACD, F–ADK,* and *F–DGK.* Since *F–ACD* is one of the required pyramids, it remains to prove *F–ADK* ≎ *K–ACD* and *F–DGK* ≎ *G–ACD.*	1. § 611.
2. *CF* ∥ plane *AG.*	2. § 646.
3. ∴ the altitude of pyramid *F–ADK* = the altitude of pyramid *C–ADK.*	3. § 664.
4. ∴ *F–ADK* ≎ *C–ADK.*	4. § 806.
5. But in *C–ADK, ACD* may be taken as base and *K* as vertex.	5. §§ 749, 750.
6. ∴ *F–ADK* ≎ *K–ACD.*	6. § 309.
7. Likewise *F–DGK* ≎ *C–DGK* = *K–CDG*; and *K–CDG* ≎ *A–CDG* = *G–ACD.*	7. By steps similar to 3–7.
8. ∴ *F–DGK* ≎ *G–ACD.*	8. § 54, 1.
9. ∴ *ACD – FGK* ≎ *F–ACD* + *G–ACD* + *K–ACD.* Q.E.D.	9. § 309.

817. Cor. I. *The volume of a truncated right triangular prism is equal to one third the product of its base and the sum of its lateral edges.*

818. Cor. II. *The volume of any truncated triangular prism is equal to one third the product of a right section and the sum of its lateral edges.*



HINT. Rt. section ACD divides truncated triangular prism QR into two truncated *right* triangular prisms.

Ex. 1334. The base of a truncated right triangular prism has for its sides 13, 14, and 15 inches; its lateral edges are 8, 11, and 13 inches. Find its volume.

Ex. 1335. In the formula of § 815: (1) put $b = 0$ and compare result with formula of § 805; (2) put $b = B$ and compare result with formula of § 799.

Ex. 1336. A frustum of a square pyramid has an altitude of 13 inches; the edges of the bases are $2\frac{1}{2}$ inches and 4 inches, respectively. Find the volume.

Ex. 1337. The edges of the bases of a frustum of a square pyramid are 3 inches and 5 inches, respectively, and the volume of the frustum is $204\frac{1}{4}$ cubic inches. Find the altitude of the frustum.

Ex. 1338. The base of a pyramid contains 144 square inches, and its altitude is 10 inches. A section of the pyramid parallel to the base divides the altitude into two equal parts. Find: (a) the area of the section; (b) the volume of the frustum formed.

Ex. 1339. A section of a pyramid parallel to the base cuts off a pyramid similar to the given pyramid.

Ex. 1340. The total areas of two similar tetrahedrons are to each other as the squares of any two homologous edges.

Ex. 1341. The altitude of a pyramid is 6 inches. A plane parallel to the base cuts the pyramid into two equivalent parts. Find the altitude of the frustum thus formed.

Ex. 1342. Two wheat bins are similar in shape; the one holds 1000 bushels, and the other 800 bushels. If the first is 15 feet deep, how deep is the second?

Ex 1343. A plane is passed parallel to the base of a pyramid cutting the altitude into two equal parts. Find: (a) the ratio of the section to the base; (b) the ratio of the pyramid cut off to the whole pyramid.

MISCELLANEOUS EXERCISES

Ex. 1344. Find the locus of all points equidistant from the three edges of a trihedral angle.

Ex. 1345. Find the locus of all points equidistant from the three faces of a trihedral angle.

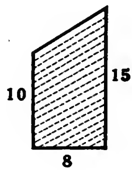
Ex. 1346. (*a*) Find the ratio of the volumes and the ratio of the total areas of two similar tetrahedrons whose homologous edges are in the ratio of 2 to 5. (*b*) Find the ratio of their homologous edges and the ratio of their total areas if their volumes are in the ratio of 1 to 27.

Ex. 1347. (*a*) Construct three or more equivalent pyramids on the same base. (*b*) Find the locus of the vertices of all pyramids equivalent to a given pyramid and standing on the same base.

Hint. Compare with Exs. 821 and 822.

Ex. 1348. The altitude of a pyramid is 12 inches. Its base is a regular hexagon whose side is 5 inches. Find the area of a section parallel to the base and 4 inches from the base ; 4 inches from the vertex.

Ex. 1349. A farmer has a corncrib 20 feet long, a cross section of which is represented in the figure, the numbers denoting feet. If the crib is entirely filled with corn in the ear, how many bushels of corn will it contain, counting 2 bushels of corn in the ear for 1 bushel of shelled corn. (Use the approximation, 1 bushel = $1\frac{1}{4}$ cubic feet. For the exact volume of a bushel, see Ex. 1439.)



Ex. 1350. A wheat elevator in the form of a frustum of a square pyramid is 30 feet high ; the edges of its bases are 12 feet and 6 feet, respectively. How many bushels of wheat will it hold ? (Use the approximation given in Ex. 1349.)

Ex. 1351. A frustum of a regular square pyramid has an altitude of 12 inches, and the edges of its bases are 4 inches and 10 inches, respectively. Find the volume of the pyramid of which the frustum is a part.

Ex. 1352. In a frustum of a regular quadrangular pyramid, the sides of the bases are 10 and 6, respectively, and the slant height is 14. Find the volume.

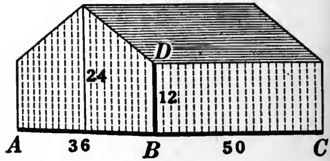
Ex. 1353. Find the lateral area of a regular triangular pyramid whose altitude is 8 inches, and each side of whose base is 6 inches.

Ex. 1354. The edge of a cube is a. Find the edge of a cube 3 times as large ; n times as large.

Ex. 1355. A berry box supposed to contain a quart of berries is in the form of a frustum of a pyramid 5 inches square at the top, $4\frac{1}{2}$ inches square at the bottom, and $2\frac{7}{8}$ inches deep. The United States dry quart contains 67.2 cubic inches. Does the box contain more or less than a quart?

Ex. 1356. The space left in a basement for a coal bin is a rectangle 8×10 feet. How deep must the bin be made to hold 10 tons of coal?

Ex. 1357. The figure represents a barn, the numbers denoting the dimensions in feet. Find the number of cubic feet in the barn.



Ex. 1358. Let AB, BC, and BD, the dimensions of the barn in Ex. 1357, be denoted by a, b, and c, respectively. Substitute the values of a, b, and c in Ahmes' formula given in § 777. Compare your result with the result obtained in Ex. 1357. Would Ahmes' formula have been correct if the Egyptian barns had been similar in shape to the barn in Ex. 1357?

Ex. 1359. How much will it cost to paint the barn in Ex. 1357 at 1 cent per square foot for lateral surfaces and 2 cents per square foot for the roof?

Ex. 1360. The barn in Ex. 1357 has a stone foundation 18 inches wide and 3 feet deep. Find the number of cubic feet of masonry if the outer surfaces of the walls are in the same planes as the sides of the barn.

Ex. 1361. The volume of a regular tetrahedron is $\frac{16}{3}\sqrt{2}$. Find its edge, slant height, and altitude.

Ex. 1362. The edge of a regular octahedron is a. Prove that the volume equals $\frac{a^3}{3}\sqrt{2}$.

Ex. 1363. The planes determined by the diagonals of a parallelopiped divide the parallelopiped into six equivalent pyramids.

Ex. 1364. A dam across a stream is 40 feet long, 12 feet high, 7 feet wide at the bottom, and 4 feet wide at the top. How many cubic feet of material are there in the dam? how many loads, counting 1 cubic yard to a load? Give the name of the geometrical solid represented by the dam.

Ex. 1365. Given S the lateral area, and H the altitude, of a regular square pyramid, find the volume.

Ex. 1366. Find the volume V, of a regular square pyramid, if its total surface is T, and one edge of the base is a.

BOOK VIII

CYLINDERS AND CONES

CYLINDERS

819. Def. A **cylindrical surface** is a surface generated by a moving straight line that continually intersects a fixed curve and remains parallel to a fixed straight line not coplanar with the given curve.

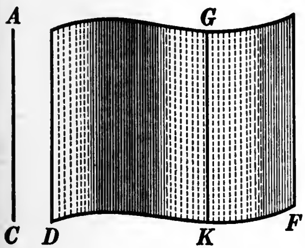


FIG. 1. Cylindrical Surface

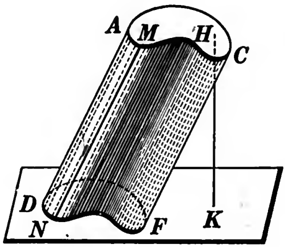


FIG. 2. Cylinder

820. Defs. By referring to §§ 693 and 694, the student may give the definitions of **generatrix, directrix,** and **element** of a cylindrical surface. Point these out in the figure.

The student should note that by changing the directrix from a broken line to a curved line, a prismatic surface becomes a cylindrical surface.

821. Def. A **cylinder** is a solid closed figure whose boundary consists of a cylindrical surface and two parallel planes cutting the generatrix in each of its positions, as *DC*.

822. Defs. The two parallel plane sections are called the **bases** of the cylinder, as *AC* and *DF* (Fig. 4); the portion of the cylindrical surface between the bases is the **lateral surface** of the cylinder; and the portion of an element of the cylindrical surface included between the bases is an **element** of the cylinder, as *MN*.

823. Def. A **right cylinder** is a cylinder whose elements are perpendicular to the bases.

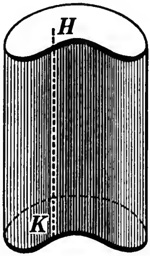


FIG. 3. Right Cylinder

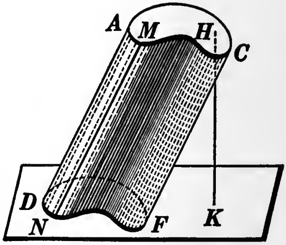


FIG. 4. Oblique Cylinder

824. Def. An **oblique cylinder** is a cylinder whose elements are not perpendicular to the bases.

825. Def. The **altitude** of a cylinder is the perpendicular from any point in the plane of one base to the plane of the other base, as *HK* in Figs. 3 and 4.

826. The following are some of the properties of a cylinder; the student should prove the correctness of each:

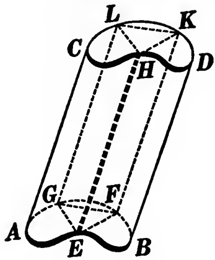
(*a*) *Any two elements of a cylinder are parallel and equal* (§§ *618 and 634*).

(*b*) *Any element of a right cylinder is equal to its altitude.*

(*c*) *A line drawn through any point in the lateral surface of a cylinder parallel to an element, and limited by the bases, is itself an element* (§§ *822 and 179*).

PROPOSITION I. THEOREM

827. *The bases of a cylinder are equal.*



Given cylinder AD with bases AB and CD.
To prove base AB = base CD.

ARGUMENT ONLY

1. Through any three points in the perimeter of base AB, as E, F, and G, draw elements EH, FK, and GL.
2. Draw EF, FG, GE, HK, KL, and LH.
3. EH is $\parallel$ and $= FK$; $\therefore EK$ is a $\square$.
4. $\therefore EF = HK$; likewise $FG = KL$ and $GE = LH$.
5. $\therefore \triangle EFG = \triangle HKL$.
6. $\therefore$ base AB may be placed upon base CD so that E, F, and G will fall upon H, K, and L, respectively.
7. But E, F, and G are *any* three points in the perimeter of base AB; *i.e. every* point in the perimeter of base AB will fall upon a corresponding point in the perimeter of base CD.
8. Likewise it can be shown that every point in the perimeter of base CD will fall upon a corresponding point in the perimeter of base AB.
9. $\therefore$ base AB may be made to coincide with base CD.
10. $\therefore$ base AB = base CD. Q.E.D.

828. Cor. I. *The sections of a cylinder made by two parallel planes cutting all the elements are equal.*

829. Cor. II. *Every section of a cylinder made by a plane parallel to its base is equal to the base.*

Ex. 1367. Every section of a cylinder made by a plane parallel to its base is a circle, if the base is a circle.

Ex. 1368. If a line joins the centers of the bases of a cylinder, this line passes through the center of every section of the cylinder parallel to the bases, if the bases are circles.

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830. Def. A **right section** of a cylinder is a section formed by a plane perpendicular to an element, as section *EF*.

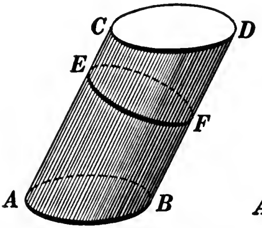


FIG. 1. Cylinder with Circular Base *AB*

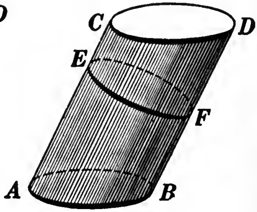


FIG. 2. Circular Cylinder

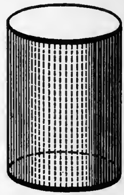


FIG. 3. Right Circular Cylinder

831. Def. A **circular cylinder** is a cylinder in which a right section is a circle; thus, in Fig. 2, if rt. section *EF* is a ⊙, cylinder *AD* is a circular cylinder.

832. Def. A **right circular cylinder** is a right cylinder whose base is a circle (Fig. 3).

833. Questions. In Fig. 1, is rt. section *EF* a ⊙? In Fig. 2, is base *AB* a ⊙? In Fig. 3, would a rt. section be a ⊙?

834. Note. The theorems and exercises on the cylinder that follow will be limited to cases in which the bases of the cylinders are circles. When the term *cylinder* is used, therefore, it must be understood to mean a *cylinder with circular bases*. See also § 846.

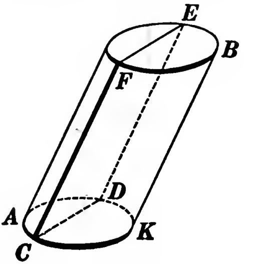
Ex. 1369. Find the locus of all points at a distance of 6 inches from a straight line 2 feet long.

Ex. 1370. Find the locus of all points: (*a*) 2 inches from the lateral surface of a right circular cylinder whose altitude is 12 inches and the radius of whose base is 5 inches; (*b*) 2 inches from the entire surface.

Ex. 1371. A log is 20 feet long and 30 inches in diameter at the smaller end. Find the dimensions of the largest piece of square timber, the same size at each end, that can be cut from the log.

PROPOSITION II. THEOREM

835. *Every section of a cylinder made by a plane passing through an element is a parallelogram.* (See § 834.)



Given cylinder AB with base AK, and $CDEF$ a section made by a plane through element CF and some point, as D, not in CF, but in the circumference of the base.

To prove $CDEF$ a ▱.

ARGUMENT	REASONS
1. Through D draw a line in plane $DF \parallel CF$.	1. § 179.
2. Then the line so drawn is an element; *i.e.* it lies in the cylindrical surface.	2. § 826, *c*.
3. But this line lies also in plane DF.	3. Arg. 1.
4. ∴ it is the intersection of plane DF with the cylindrical surface, and coincides with DE.	4. § 614.
5. ∴ DE is a str. line and is $\parallel$ and $=$ CF.	5. § 826, *a*.
6. Also CD and EF are str. lines.	6. § 616.
7. ∴ $CDEF$ is a ▱. Q.E.D.	7. § 240.

836. Cor. *Every section of a right circular cylinder made by a plane passing through an element is a rectangle.*

Ex. 1372. In the figure of Prop. II, the radius of the base is 4 inches, element CF is 12 inches, CD is 1 inch from the center of the base, and CF makes with CD an angle of 60°. Find the area of section $CDEF$.

Ex. 1373. Every section of a cylinder, parallel to an element, is a parallelogram. How is the base of this cylinder restricted? (See § 834.)

CONES

837. Def. A **conical surface** is a surface generated by a moving straight line that continually intersects a fixed curve and that passes through a fixed point not in the plane of the curve.

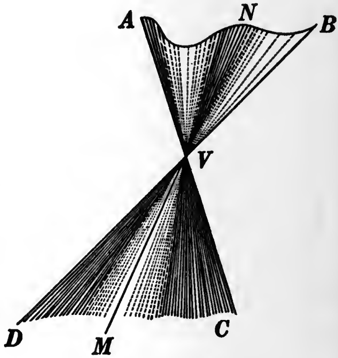


FIG. 1. Conical Surface

838. Defs. By referring to §§ 693, 694, and 746, the student may give the definitions of **generatrix, directrix, vertex, element,** and **upper and lower nappes** of a conical surface. Point these out in the figure.

The student should observe that by changing the directrix from a broken line to a curved line, a pyramidal surface becomes a conical surface.

839. Def. A **cone** is a solid closed figure whose boundary consists of the portion of a conical surface extending from its vertex to a plane cutting all its elements, and the section formed by this plane.

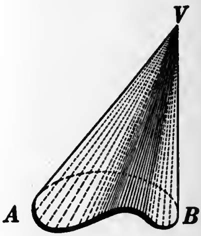


FIG. 2. Cone

840. Defs. By referring to §§ 748 and 822, the student may give the definitions of **vertex, base, lateral surface,** and **element** of a cone. Point these out in the figure.

841. Def. A **circular cone** is a cone containing a circular section such that a line joining the vertex of the cone to the center of the section is perpendicular to the section.

Thus in Fig. 4, if section *AB* of cone *V–CD* is a ⊙ with center *O*, such that *VO* is ⊥ the section, cone *V–CD* is a circular cone.

842. Def. The **altitude** of a cone is the perpendicular from its vertex to the plane of its base, as *VC* in Fig. 3 and *VO* in Fig. 5.

843. Defs. In a cone with a circular base, if the line joining its vertex to the center of its base is perpendicular to the

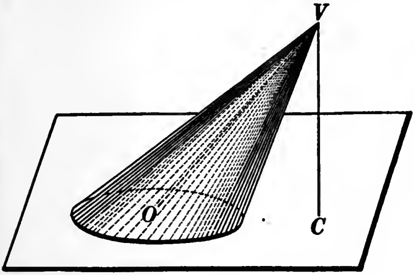


FIG. 3. Cone with Circular Base

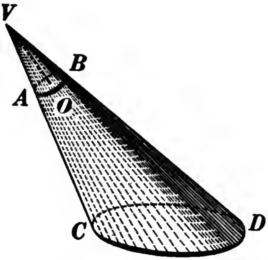


FIG. 4. Circular Cone

plane of the base, the cone is a **right circular cone** (Fig. 5).

If such a line is not perpendicular to the plane of the base, the cone is called an **oblique cone** (Fig. 3).

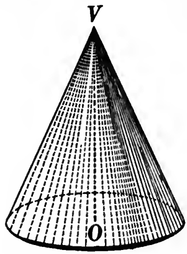


FIG. 5. Right Circular Cone

844. Def. The **axis** of a right circular cone is the line joining its vertex to the center of its base, as VO, Fig. 5.

845. The following are some of the properties of a cone; the student should prove the correctness of each:

(*a*) *The elements of a right circular cone are equal.*

(*b*) *The axis of a right circular cone is equal to its altitude.*

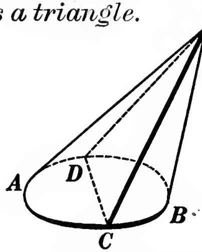
(*c*) *A straight line drawn from the vertex of a cone to any point in the perimeter of its base is an element.*

846. Note. The theorems and exercises on the cone that follow will be limited to cases in which the bases of the cones are circles, though not necessarily to *circular cones.* When the term *cone* is used, therefore, it must be understood to mean a *cone with circular base.* See also § 834.

Ex. 1374. What is the locus of all points 2 inches from the lateral surface, and 2 inches from the base, of a right circular cone whose altitude is 12 inches and the radius of whose base is 5 inches?

PROPOSITION III. THEOREM

847. *Every section of a cone made by a plane passing through its vertex is a triangle.*



Given cone $V\text{–}AB$ with base AB and section VCD made by a plane through V.

To prove VCD a $\triangle$.

ARGUMENT	REASONS
1. From V draw str. lines to C and D.	1. § 54, 15.
2. Then the lines so drawn are elements; *i.e.* they lie in the conical surface.	2. § 845, *c*.
3. But these lines lie also in plane VCD.	3. § 603, *a*.
4. $\therefore$ they are the intersections of plane VCD with the conical surface, and coincide with VC and VD, respectively.	4. § 614.
5. Also CD is a str. line.	5. § 616.
6. $\therefore$ VC, VD, and CD are str. lines and VCD is a $\triangle$. Q.E.D.	6. § 92.

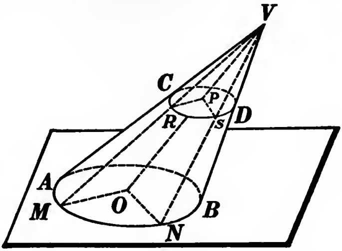
Ex. 1375. What kind of triangle in general is the section of a cone through the vertex, if the cone is oblique? if the cone is a right circular cone? Can any section of an oblique cone be perpendicular to the base of the cone? of a right circular cone? Explain.

Ex. 1376. Find the locus of all straight lines making a given angle with a given straight line, at a given point in the line. What will this locus be if the given angle is 90°?

Ex. 1377. Find the locus of all straight lines making a given angle with a given plane at a given point. What will the locus be if the given angle is 90°?

Proposition IV. Theorem

848. *Every section of a cone made by a plane parallel to its base is a circle.* (See § 846.)



Given CD a section of cone $V\text{–}AB$ made by a plane $\parallel$ base AB.

To prove section CD a $\odot$.

Outline of Proof

1. Let R and S be any two points on the boundary of section CD; pass planes through OV and points R and S.
2. Prove $\triangle VOM \sim \triangle VPR$ and $\triangle VON \sim \triangle VPS$.
3. Then $\dfrac{OM}{PR} = \dfrac{VO}{VP}$ and $\dfrac{ON}{PS} = \dfrac{VO}{VP}$; *i.e.* $\dfrac{OM}{PR} = \dfrac{ON}{PS}$.
4. But $OM = ON$; $\therefore PS = PR$; *i.e.* P is equidistant from any two points on the boundary of section CD.
5. $\therefore$ section CD is a $\odot$. Q.E.D.

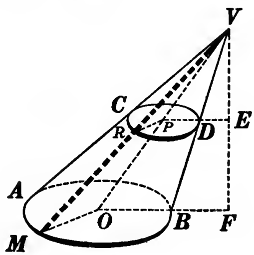
849. Cor. *Any section of a cone parallel to its base is to the base as the square of its distance from the vertex is to the square of the altitude of the cone.*

Outline of Proof

By § 563, $\dfrac{\text{section } CD}{\text{base } AB} = \dfrac{\overline{PR}^2}{\overline{OM}^2}$.

Prove $\dfrac{PR}{OM} = \dfrac{VP}{VO} = \dfrac{VE}{VF}$.

Then $\dfrac{\text{section } CD}{\text{base } AB} = \dfrac{\overline{VE}^2}{\overline{VF}^2}$.



For applications of §§ 848 and 849, see Exs. 1380–1384.

MENSURATION OF THE CYLINDER AND CONE

Areas

850. Def. A **plane** is **tangent to a cylinder** if it contains an element, but no other point, of the cylinder.

851. Def. A **prism** is **inscribed in a cylinder** if its lateral edges are elements of the cylinder, and the bases of the two figures lie in the same plane.

852. Def. A **prism** is **circumscribed about a cylinder** if its lateral faces are all tangent to the cylinder, and the bases of the two figures lie in the same plane.

Ex. 1378. How many planes can be tangent to a cylinder? If two of these planes intersect, the line of intersection is parallel to an element. How are the bases of the cylinders in §§ 850–852 restricted? (See § 834.)

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853. Before proceeding further it might be well for the student to review the more important steps in the development of the area of a circle. In that development it was shown that:

(1) The area of a regular polygon circumscribed about a circle is greater, and the area of a regular polygon inscribed in a circle less, than the area of the regular circumscribed or inscribed polygon of twice as many sides (§ 541).

(2) By repeatedly doubling the number of sides of regular circumscribed and inscribed polygons of the same number of sides, and making the polygons always regular, their areas approach a common limit (§ 546).

(3) This common limit is defined as the area of the circle (§ 558).

(4) Finally follows the theorem for the area of the circle (§ 559).

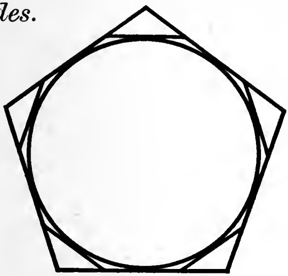
It will be observed that precisely the same method is used throughout the mensuration of the cylinder and the cone.

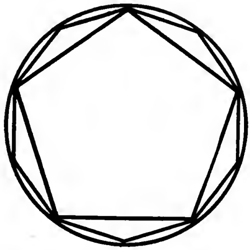
Compare carefully the four articles just cited with §§ 854, 855, 857, and 858.

Proposition V. Theorem

854. I. *The lateral area of a regular prism circumscribed about a right circular cylinder is greater than the lateral area of the regular circumscribed prism whose base has twice as many sides.*

II. *The lateral area of a regular prism inscribed in a right circular cylinder is less than the lateral area of the regular inscribed prism whose base has twice as many sides.*





The proof is left as an exercise for the student.

Hint. See § 541. Let the given figures represent the bases of the actual figures.

Ex. 1379. A regular quadrangular and a regular octangular prism are inscribed in a right circular cylinder with altitude 25 inches and radius of base 10 inches. Find the difference between their lateral areas.

Ex. 1380. The line joining the vertex of a cone to the center of the base, passes through the center of every section parallel to the base.

Ex. 1381. Sections of a cone made by planes parallel to the base are to each other as the squares of their distances from the vertex.

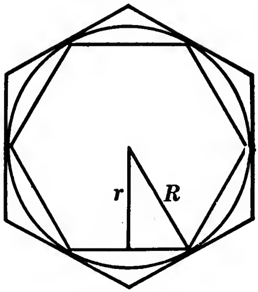
Ex. 1382. The base of a cone contains 144 square inches and the altitude is 10 inches. Find the area of a section of the cone 3 inches from the vertex; 5 inches from the vertex.

Ex. 1383. The altitude of a cone is 12 inches. How far from the vertex must a plane be passed parallel to the base so that the section shall be one half as large as the base? one third? one nth?

Ex. 1384. The altitude of a cone is 20 inches; the area of the section parallel to the base and 12 inches from the vertex is 90 square inches. Find the area of the base.

Proposition VI. Theorem

855. *By repeatedly doubling the number of sides of the bases of regular prisms circumscribed about, and inscribed in, a right circular cylinder, and making the bases always regular polygons, their lateral areas approach a common limit.*



Given H the common attitude, R and r the apothems of the bases, P and p the perimeters of the bases, and S and s the lateral areas, respectively, of regular circumscribed and inscribed prisms whose bases have the same number of sides. Let the given figure represent the base of the actual figure.

To prove that by repeatedly doubling the number of sides of the bases of the prisms, and making them always regular polygons, S and s approach a common limit.

Argument	Reasons
1. $S = P \cdot H$ and $s = p \cdot H$.	1. § 763.
2. $\therefore \frac{S}{s} = \frac{P}{p}$.	2. § 54, 8 *a*.
3. But $\frac{P}{p} = \frac{R}{r}$.	3. § 538.
4. $\therefore \frac{S}{s} = \frac{R}{r}$.	4. § 54, 1.
5. $\therefore \frac{S - s}{S} = \frac{R - r}{R}$.	5. § 399.

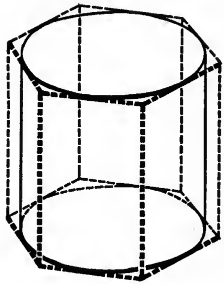
Argument	Reasons
6. $\therefore S - s = S\,\dfrac{R - r}{R}$.	6. § 54, 7 *a*.
7. But by repeatedly doubling the number of sides of the bases of the prisms, and making them always regular polygons, $R - r$ can be made less than any previously assigned value, however small.	7. § 543, I.
8. $\therefore \dfrac{R - r}{R}$ can be made less than any previously assigned value, however small.	8. § 586.
9. $\therefore S\,\dfrac{R - r}{R}$ can be made less than any previously assigned value, however small, S being a decreasing variable.	9. § 587.
10. $\therefore S - s$, being always equal to $S\,\dfrac{R - r}{R}$, can be made less than any previously assigned value, however small.	10. § 309.
11. $\therefore S$ and s approach a common limit. Q.E.D.	11. § 594.

856. Note. The above proof is limited to *regular* prisms, but it can be shown that the limit of the lateral area of any inscribed (or circumscribed) prism is the same by whatever method the number of the sides of its base is successively increased, provided that each side approaches zero as a limit. (See also § 549.) Compare the proof of § 855 with that of § 546, I.

857. Def. The **lateral area of a right circular cylinder** is the common limit which the successive lateral areas of circumscribed and inscribed regular prisms (having bases containing 3, 4, 5, etc., sides) approach as the number of sides of the bases is successively increased and each side approaches zero as a limit.

Proposition VII. Theorem

858. *The lateral area of a right circular cylinder is equal to the product of the circumference of its base and its altitude.*



Given a rt. circular cylinder with its lateral area denoted by S, the circumference of its base by C, and its altitude by H.

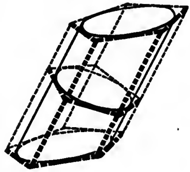
To prove $S = C \cdot H$.

Argument	Reasons
1. Circumscribe about the rt. circular cylinder a regular prism. Denote its lateral area by S', the perimeter of its base by P, and its altitude by H.	1. § 852.
2. Then $S' = P \cdot H$.	2. § 763.
3. As the number of sides of the base of the regular circumscribed prism is repeatedly doubled, P approaches C as a limit.	3. § 550.
4. $\therefore P \cdot H$ approaches $C \cdot H$ as a limit.	4. § 590.
5. Also S' approaches S as a limit.	5. § 857.
6. But S' is always equal to $P \cdot H$.	6. Arg. 2.
7. $\therefore S = C \cdot H$. Q.E.D.	7. § 355.

859. Cor. *If S denotes the lateral area, T the total area, H the altitude, and R the radius of the base, of a right circular cylinder,*

$$S = 2\,\pi RH;$$
$$T = 2\,\pi RH + 2\,\pi R^2 = 2\,\pi R(H + R).$$

860. Note. Since the lateral area of an oblique prism is equal to the product of the perimeter of a right section and a lateral edge (§ 762), the student would naturally infer that the lateral area of an oblique cylinder with circular bases is equal to the product of the perimeter of a right section and an element. This statement is true. But the right section of an oblique cylinder with circular base is not a circle. And since the only curve dealt with in elementary geometry is the circle, this theorem and its applications have been omitted here.



Ex. 1385. Find the lateral area and total area of a right circular cylinder whose altitude is 20 centimeters and the diameter of whose base is 10 centimeters.

Ex. 1386. How many square inches of tin will be required to make an open cylindrical pail 8 inches in diameter and 10 inches deep, making no allowance for waste?

Ex. 1387. In a right circular cylinder, find the ratio of the lateral area to the sum of the two bases. What is this ratio if the altitude and the radius of base are equal?

Ex. 1388. Find the altitude of a right circular cylinder if its lateral area is S and the radius of its base R.

Ex. 1389. Find the radius of the base of a right circular cylinder if its total area is T and its altitude is H.

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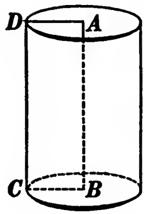
861. Def. Because it may be generated by a rectangle revolving about one of its sides as an axis, a right circular cylinder is sometimes called a **cylinder of revolution.**

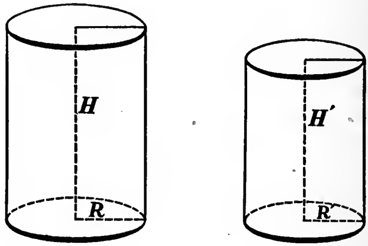


862. Questions. What part of the cylinder will side CD, opposite the axis AB, generate? What will AD and BC generate? What will the plane AC generate? What might CD in any one of its positions be called?

863. Def. Similar cylinders of revolution are cylinders generated by similar rectangles revolving about homologous sides as axes.

PROPOSITION VIII. THEOREM

864. *The lateral areas, and the total areas, of two similar cylinders of revolution are to each other as the squares of their altitudes, and as the squares of the radii of their bases.*



Given two similar cylinders of revolution with their lateral areas denoted by S and S', their total areas by T and T', their altitudes by H and H', and the radii of their bases by R and R', respectively.

To prove: (*a*) $\frac{S}{S'}=\frac{H^2}{H'^2}=\frac{R^2}{R'^2}$.

(*b*) $\frac{T}{T'}=\frac{H^2}{H'^2}=\frac{R^2}{R'^2}$.

ARGUMENT	REASONS
1. $S=2\pi RH$ and $S'=2\pi R'H'$.	1. § 859.
2. $\therefore \frac{S}{S'}=\frac{2\pi RH}{2\pi R'H'}=\frac{RH}{R'H'}=\frac{R}{R'}\cdot\frac{H}{H'}$.	2. § 54, 8 *a*.
3. But rectangle $RH \sim$ rectangle $R'H'$.	3. § 863.
4. $\therefore \frac{H}{H'}=\frac{R}{R'}$.	4. § 419.
5. $\therefore \frac{S}{S'}=\frac{H}{H'}\cdot\frac{H}{H'}=\frac{H^2}{H'^2}$.	5. § 309.
6. Also $\frac{S}{S'}=\frac{R}{R'}\cdot\frac{R}{R'}=\frac{R^2}{R'^2}$.	6. § 309.
7. Again, $T=2\pi R(H+R)$ and $T'=2\pi R'(H'+R')$.	7. § 859.

ARGUMENT	REASONS
8. $\therefore \frac{T}{T'}=\frac{R(H+R)}{R'(H'+R')}=\frac{R}{R'}\cdot\frac{H+R}{H'+R'}.$	8. § 54, 8 *a*.
9. But, from Arg. 4, $\frac{H+R}{H'+R'}=\frac{H}{H'}=\frac{R}{R'}.$	9. § 401.
10. $\therefore \frac{T}{T'}=\frac{H}{H'}\cdot\frac{H}{H'}=\frac{H^2}{H'^2}.$	10. § 309.
11. Also $\frac{T}{T'}=\frac{R}{R'}\cdot\frac{R}{R'}=\frac{R^2}{R'^2}.$ Q.E.D.	11. § 309.

Ex. 1390. The altitudes of two similar cylinders of revolution are 5 inches and 7 inches, respectively, and the total area of the first is 675 square inches. Find the total area of the second.

Ex. 1391. The lateral areas of two similar cylinders of revolution are 320 square inches and 500 square inches, and the radius of the base of the larger is 10 inches. Find the radius of the base of the smaller.

Ex. 1392. Two adjacent sides of a rectangle are a and b; find the lateral area of the cylinder generated by revolving the rectangle: (1) about a as an axis; (2) about b as an axis. Put the results in the form of a general statement. Have you proved this general statement?

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865. Def. The **slant height** of a right circular cone is a straight line joining its vertex to any point in the circumference of its base. Thus any element of such a cone is its slant height.

866. Def. A **plane** is **tangent to a cone** if it contains an element, but no other point, of the cone.

867. Def. A **pyramid** is **inscribed in a cone** if its base is inscribed in the base of the cone and its vertex coincides with the vertex of the cone.

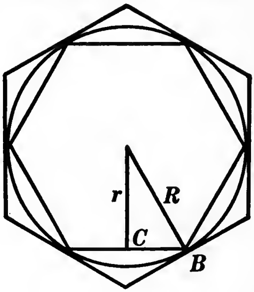
868. Def. A **pyramid** is **circumscribed about a cone** if its base is circumscribed about the base of the cone and its vertex coincides with the vertex of the cone.

869. The student may state and prove the theorems on the right circular cone corresponding to those mentioned in § 854.

Ex. 1393. How many planes can be tangent to a cone? Through what point must each of these planes pass? Prove. How are the bases of the cones in §§ 866–868 restricted? (See § 846.)

Proposition IX. Theorem

870. *By repeatedly doubling the number of sides of the bases of regular pyramids circumscribed about, and inscribed in, a right circular cone, and making the bases always regular polygons, their lateral areas approach a common limit.*



Given H the common altitude, L and l the slant heights, R and r the apothems of the bases, P and p the perimeters of the bases, and S and s the lateral areas, respectively, of regular circumscribed and inscribed pyramids whose bases have the same number of sides. Let the given figure represent the base of the actual figure.

To prove that by repeatedly doubling the number of sides of the bases of the pyramids, and making them always regular polygons, S and s approach a common limit.

Argument	Reasons
1. $S = \frac{1}{2}PL$ and $s = \frac{1}{2}pl$.	1. § 766.
2. $\therefore \frac{S}{s} = \frac{PL}{pl} = \frac{P}{p} \cdot \frac{L}{l}$.	2. § 54, 8 *a*.
3. But $\frac{P}{p} = \frac{R}{r}$.	3. § 538.
4. $\therefore \frac{S}{s} = \frac{R}{r} \cdot \frac{L}{l} = \frac{RL}{rl}$.	4. § 309.
5. $\therefore \frac{S - s}{S} = \frac{RL - rl}{RL}$.	5. § 399.

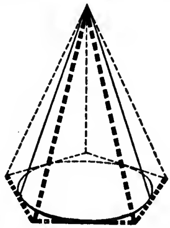
Argument	Reasons
6. $\therefore S - s = S\,\dfrac{RL - rl}{RL}$.	6. 54, 7 *a*.
7. Now $L - l < CB$.	7. § 168.
8. But by repeatedly doubling the number of sides of the bases of the pyramids, and making them always regular polygons, CB can be made less than any previously assigned value, however small.	8. Arg. 3, § 543, I.
9. $\therefore L - l$, being always less than CB, can be made less than any previously assigned value, however small.	9. § 54, 10.
10. $\therefore$ the limit of $l = L$.	10. § 349.
11. Also the limit of $r = R$.	11. § 543, I.
12. $\therefore$ the limit of $rl = RL$.	12. § 592.
13. $\therefore RL - rl$ can be made less than any previously assigned value, however small.	13. § 349.
14. $\therefore \dfrac{RL - rl}{RL}$ can be made less than any previously assigned value, however small.	14. § 586.
15. Similar to Arg. 9, § 855.	15. § 587.
16. Similar to Arg. 10, § 855.	16. § 309.
17. $\therefore S$ and s approach a common limit. Q.E.D.	17. § 594.

871. Note. The proof of § 870 is limited in the same manner as the proof of § 855. Read § 856.

872. Def. The **lateral area of a right circular cone** is the common limit which the successive lateral areas of circumscribed and inscribed regular pyramids approach as the number of sides of the bases is successively increased and each side approaches zero as a limit.

Proposition X. Theorem

873. *The lateral area of a right circular cone is equal to one half the product of the circumference of its base and its slant height.*



Given a rt. circular cone with its lateral area denoted by S, the circumference of its base by C, and its slant height by L.

To prove $S = \frac{1}{2} C \cdot L$.

The proof is left as an exercise for the student.

874. Question. What changes are necessary in the proof of Prop. VII to make it the proof of Prop. X?

875. Cor. *If S denotes the lateral area, T the total area, L the slant height, and R the radius of the base, of a right circular cone,*

$$S = \pi RL;$$
$$T = \pi RL + \pi R^2 = \pi R(L + R).$$

Ex. 1394. The altitude of a right circular cone is 12 inches and the radius of the base 8 inches. Find the lateral area and the total area of the cone.

Ex. 1395. How many yards of canvas 30 inches wide will be required to make a conical tent 16 feet high and 20 feet in diameter, if 10% of the goods is allowed for cutting and fitting?

Ex. 1396. The lateral area of a right circular cone is $\frac{440}{7}\sqrt{89}$ square inches, and the radius of the base is 10 inches. Find the altitude.

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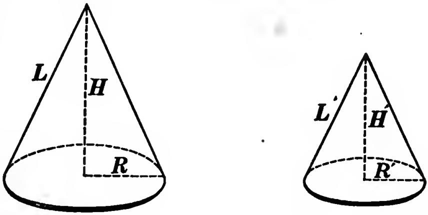
876. Def. Because it may be generated by a right triangle revolving about one of its sides as an axis, a right circular cone is sometimes called a **cone of revolution.**

877. Def. **Similar cones of revolution** are cones generated by similar right triangles revolving about homologous sides as axes.

Proposition XI. Theorem

878. *The lateral areas, and the total areas, of two similar cones of revolution are to each other as the squares of their altitudes, as the squares of their slant heights, and as the squares of the radii of their bases.*



Given two similar cones of revolution with their lateral areas denoted by S and S', their total areas by T and T', their altitudes by H and H', their slant heights by L and L', and the radii of their bases by R and R', respectively.

To prove: (a) $\dfrac{S}{S'} = \dfrac{H^2}{H'^2} = \dfrac{L^2}{L'^2} = \dfrac{R^2}{R'^2}$.

(b) $\dfrac{T}{T'} = \dfrac{H^2}{H'^2} = \dfrac{L^2}{L'^2} = \dfrac{R^2}{R'^2}$.

The proof is left as an exercise for the student.

Hint. Apply the method of proof used in Prop. VIII.

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879. Def. A **frustum of a cone** is the portion of the cone included between the base and a section of the cone made by a plane parallel to the base.

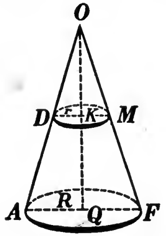
880. Questions. What are the upper and lower bases of a frustum of a cone? the altitude? What kind of a figure is the upper base of a frustum of a right circular cone (§ 848)?

881. Def. The **slant height** of a frustum of a right circular cone is the length of that portion of an element of the cone included between the bases of the frustum.

Ex. 1397. Every section of a frustum of a cone, made by a plane passing through an element, is a trapezoid.

Proposition XII. Theorem

882. *The lateral area of a frustum of a right circular cone is equal to one half the product of the sum of the circumferences of its bases and its slant height.*



Given frustum AM, of right circular cone $O\text{–}AF$, with its lateral area denoted by S, the circumferences of its bases by C and c, the radii of its bases by R and r, and its slant height by L.

To prove $S = \frac{1}{2}(C + c)L$.

Argument	Reasons
1. $S =$ lateral area of cone $O\text{–}AF$ minus lateral area of cone $O\text{–}DM$.	1. § 54, 11.
2. Let L' denote the slant height of cone $O\text{–}DM$. Then $S = \frac{1}{2}C(L + L') - \frac{1}{2}cL'$ $= \frac{1}{2}CL + \frac{1}{2}L'(C - c)$.	2. § 873.
It now remains to find the value of L'.	
3. $\frac{C}{c} = \frac{R}{r}$.	3. § 556.
4. But $\triangle OKD \sim \triangle OQA$.	4. § 422.
5. $\therefore \frac{R}{r} = \frac{OA}{OD} = \frac{L + L'}{L'}$.	5. § 424, 2.
6. $\therefore \frac{C}{c} = \frac{L + L'}{L'}$.	6. § 54, 1.
7. $\therefore L' = \frac{cL}{C - c}$.	7. Solving for L'.
8. $\therefore S = \frac{1}{2}CL + \frac{1}{2} \cdot \frac{cL}{C-c}(C - c) = \frac{1}{2}(C + c)L$. Q.E.D.	8. § 309.

883. Cor. I. *If* S *denotes the lateral area,* T *the total area,* L *the slant height, and* R *and* r *the radii of the bases, of a frustum of a right circular cone,*

$$S = \pi L(R + r);$$
$$T = \pi L(R + r) + \pi(R^2 + r^2).$$

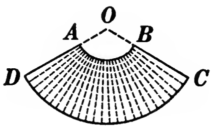
884. Cor. II. *The lateral area of a frustum of a right circular cone is equal to the product of its slant height and the circumference of a section midway between its bases.*

Ex. 1398. If S denotes the lateral area, L the slant height, and C the circumference of a section midway between the bases, of a frustum of a right circular cone, then $S = CL$.

Ex. 1399. In the formulas of § 883: (a) make $r = 0$ and compare results with formulas of § 875; (b) make $b = B$ and compare results with formulas of § 859.

Ex. 1400. The altitude of a frustum of a right circular cone is 16 inches, and the diameters of its bases are 20 inches and 30 inches, respectively. Find its lateral area and also its total area.

Ex. 1401. In the figure, AB and CD are arcs of circles; $OA = 2$ inches, $OD = 5$ inches, and $\angle DOC = 120°$. Cut figure $ABCD$ out of paper and form it into a frustum of a cone. Find its lateral area and also its total area.



Ex. 1402. A frustum of a right circular cone whose altitude is 4 inches and radii of bases 4 inches and 7 inches, respectively, is made as indicated in Ex. 1401. Find the radius of the circle from which it must be cut.

Ex. 1403. The sum of the total areas of two similar cylinders of revolution is 216 square inches, and one altitude is $\frac{3}{4}$ of the other. Find the total area of each cylinder.

Ex. 1404. A regular triangular and a regular hexangular pyramid are inscribed in a right circular cone with altitude 20 inches and with radius of base 4 inches. Find the difference between their lateral areas.

Ex. 1405. Cut out of paper a semicircle whose radius is 4 inches, and find its area. Form a cone with this semicircle and find its lateral area by § 875. Do the two results agree?

Ex. 1406. The slant height, and the diameter of the base, of a right circular cone are each equal to L. Find the total area.

VOLUMES

Proposition XIII. Theorem

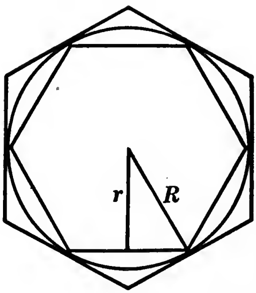
885. I. *The volume of a prism whose base is a regular polygon and which is circumscribed about a cylinder is greater than the volume of the circumscribed prism whose base is a regular polygon with twice as many sides.*

II. *The volume of a prism whose base is a regular polygon and which is inscribed in a cylinder is less than the volume of the inscribed prism whose base is a regular polygon with twice as many sides.*

The figures and proofs are left as exercises for the student.

Proposition XIV. Theorem

886. *By repeatedly doubling the number of sides of the bases of prisms circumscribed about, and inscribed in, a cylinder, and making the bases always regular polygons, their volumes approach a common limit.*



Given H the common altitude, B and b the areas of the bases, and V and v the volumes, respectively, of circumscribed and inscribed prisms whose bases are regular and have the same number of sides. Let the given figure represent the base of the actual figure.

To prove that by repeatedly doubling the number of sides of the bases of the prisms, and making them always regular polygons, V and v approach a common limit.

Argument	Reasons
1. $V = B \cdot H$ and $v = b \cdot H$.	1. § 799.
2. $\therefore \frac{V}{v} = \frac{B \cdot H}{b \cdot H} = \frac{B}{b}$.	2. § 54, 8 *a*.
3. $\therefore \frac{V - v}{V} = \frac{B - b}{B}$.	3. § 399.
4. $\therefore V - v = V\frac{B - b}{B}$.	4. § 54, 7 *a*.
5. Similar to Arg. 7, § 855.	5. § 546, II.
6. Similar to Arg. 8, § 855.	6. § 586.
7. Similar to Arg. 9, § 855.	7. § 587.
8. Similar to Arg. 10, § 855.	8. § 309.
9. $\therefore V$ and v approach a common limit. Q.E.D.	9. § 594.

887. Note. The proof of § 886 is limited in the same manner as the proof of § 855. Read § 856.

Ex. 1407. The total area of a right circular cone whose altitude is 10 inches is 280 square inches. Find the total area of the cone cut off by a plane parallel to the base and 6 inches from the base.

Ex. 1408. The altitude of a right circular cone is 12 inches. What part of the lateral surface is cut off by a plane parallel to the base and 6 inches from the vertex?

Ex. 1409. The altitude of a right circular cone is H. How far from the vertex must a plane be passed parallel to the base so that the lateral area and the total area of the cone cut off shall be one half that of the original cone? one third? one nth?

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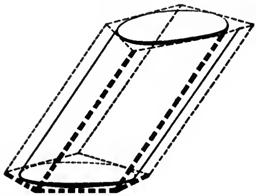
888. Def. The **volume of a cylinder** is the common limit which the successive volumes of circumscribed and inscribed prisms approach as the number of sides of the bases is successively increased, and each side approaches zero as a limit.

Proposition XV. Theorem

889. *The volume of a cylinder is equal to the product of its base and its altitude.*



Given a cylinder, with its volume denoted by V, its base by B, and its altitude by H.

To prove $V = B \cdot H$.

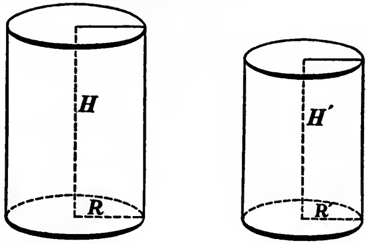
Argument	Reasons
1. Circumscribe about the cylinder a prism whose base is a regular polygon. Denote its volume by V' and the area of its base by B'.	1. § 852.
2. Then $V' = B' \cdot H$.	2. § 799.
3. As the number of sides of the base of the circumscribed prism is repeatedly doubled, B' approaches B as a limit.	3. § 558.
4. $\therefore B' \cdot H$ approaches $B \cdot H$ as a limit.	4. § 590.
5. Also V' approaches V as a limit.	5. § 888.
6. But V' is always equal to $B' \cdot H$.	6. Arg. 2.
7. $\therefore V = B \cdot H$. Q.E.D.	7. § 355.

890. Cor. *If V denotes the volume, H the altitude, and R the radius of the base, of a cylinder,*

$$V = \pi R^2 H.$$

PROPOSITION XVI. THEOREM

891. *The volumes of two similar cylinders of revolution are to each other as the cubes of their altitudes, and as the cubes of the radii of their bases.*



Given two similar cylinders of revolution with their volumes denoted by V and V', their altitudes by H and H', and the radii of their bases by R and R', respectively.

To prove $\dfrac{V}{V'} = \dfrac{H^3}{H'^3} = \dfrac{R^3}{R'^3}$.

ARGUMENT	REASONS
1. $V = \pi R^2 H$ and $V' = \pi R'^2 H'$.	1. § 890.
2. $\therefore \dfrac{V}{V'} = \dfrac{\pi R^2 H}{\pi R'^2 H'} = \dfrac{R^2 H}{R'^2 H'} = \dfrac{R^2}{R'^2} \cdot \dfrac{H}{H'}$.	2. § 54, 8 *a*.
3. But rectangle $RH \sim$ rectangle $R'H'$.	3. § 863.
4. $\therefore \dfrac{H}{H'} = \dfrac{R}{R'}$.	4. § 419.
5. $\therefore \dfrac{V}{V'} = \dfrac{R^2}{R'^2} \cdot \dfrac{R}{R'} = \dfrac{R^3}{R'^3}$.	5. § 309.
6. But, from Arg. 4, $\dfrac{H^3}{H'^3} = \dfrac{R^3}{R'^3}$.	6. § 54, 13.
7. $\therefore \dfrac{V}{V'} = \dfrac{H^3}{H'^3}$; *i.e.* $\dfrac{V}{V'} = \dfrac{H^3}{H'^3} = \dfrac{R^3}{R'^3}$. Q. E. D.	7. § 309.

Ex. 1410. The volumes of two similar cylinders of revolution are 135 cubic inches and 1715 cubic inches, respectively, and the altitude of the first is 3 inches. Find the altitude of the second.

Ex. 1411. A cylinder of revolution has an altitude of 12 inches and a base with a radius of 5 inches. Find the total area of a similar cylinder whose volume is 8 times that of the given cylinder.

Ex. 1412. The dimensions of a rectangle are 6 inches and 8 inches, respectively. Find the volume of the solid generated by revolving the rectangle: (*a*) about its longer side as an axis; (*b*) about its shorter side. Compare the ratio of these volumes with the ratio of the sides of the rectangle.

Ex. 1413. Cylinders having equal bases and equal altitudes are equivalent.

Ex. 1414. Any two cylinders are to each other as the products of their bases and their altitudes.

Ex. 1415. (*a*) Two cylinders having equal bases are to each other as their altitudes, and (*b*) having equal altitudes are to each other as their bases.

Ex. 1416. The volume of a right circular cylinder is equal to the product of its lateral area and one half the radius of its base.

Ex. 1417. Cut out a rectangular piece of paper 6×9 inches. Roll this into a right circular cylinder and find its volume (two answers).

Ex. 1418. A cistern in the form of a right circular cylinder is to be 20 feet deep and 8 feet in diameter. How much will it cost to dig it at 5 cents a cubic foot?

Ex. 1419. Find the altitude of a right circular cylinder if its volume is V and the radius of its base R.

Ex. 1420. In a certain right circular cylinder the lateral area and the volume have the same numerical value. (*a*) Find the radius of the base. (*b*) Find the volume if the altitude is equal to the diameter of the base.

Ex. 1421. A cylinder is inscribed in a cube whose edge is 10 inches. Find: (*a*) the volume of each; (*b*) the ratio of the cylinder to the cube.

Ex. 1422. A cylindrical tin tomato can is $4\frac{9}{16}$ inches high, and the diameter of its base is 4 inches. Does it hold more or less than a liquid quart, *i.e.* $\frac{231}{4}$ cubic inches?

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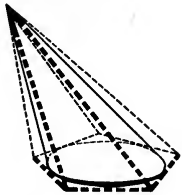
892. The student may:

(*a*) State and prove the theorems on the cone corresponding to those given in §§ 885 and 886.

(*b*) State, by aid of § 888, the definition of the **volume of a cone.**

Proposition XVII. Theorem

893. *The volume of a cone is equal to one third the product of its base and its altitude.*



Given a cone with its volume denoted by V, its base by B, and its altitude by H.

To prove $V = \frac{1}{3} B \cdot H.$

The proof is left as an exercise for the student.

894. Question. What changes must be made in the proof of Prop. XV to make it the proof of Prop. XVII?

895. Cor. *If* V *denotes the volume,* H *the altitude, and* R *the radius of the base of a cone,*

$$V = \frac{1}{3} \pi R^2 H.$$

Ex. 1423. Any two cones are to each other as the products of their bases and altitudes.

Ex. 1424. The slant height of a right circular cone is 18 inches and makes with the base an angle of 60°; the radius of the base is 8 inches. Find the volume of the cone.

Ex. 1425. The base of a cone has a radius of 12 inches; an element of the cone is 24 inches long and makes with the base an angle of 30°. Find the volume of the cone.

Ex. 1426. The hypotenuse of a right triangle is 17 inches and one side is 15 inches. Find the volume of the solid generated by revolving the triangle about its shortest side as an axis.

Ex. 1427. A cone and a cylinder have equal bases and equal altitudes. Find the ratio of their volumes.

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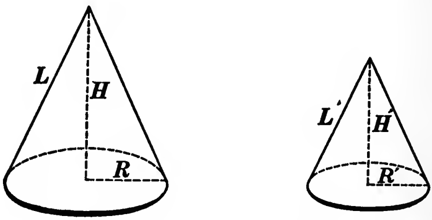
896. Historical Note. To Eudoxus is credited the proof of the proposition that "every cone is the third part of a cylinder on the same base and with the same altitude." Proofs of this proposition were also given later by Archimedes and Brahmagupta. (Compare with § 809.)

Proposition XVIII. Theorem

897. *The volumes of two similar cones of revolution are to each other as the cubes of their altitudes, as the cubes of their slant heights, and as the cubes of the radii of their bases.*



Given two similar cones of revolution with their volumes denoted by V and V', their altitudes by H and H', their slant heights by L and L', and the radii of their bases by R and R', respectively.

To prove $$\frac{V}{V'}=\frac{H^3}{H'^3}=\frac{L^3}{L'^3}=\frac{R^3}{R'^3}.$$

The proof is left as an exercise for the student.

Hint. Apply the method of proof used in Prop. XVI.

Ex. 1428. If the altitude of a cone of revolution is three fourths that of a similar cone, what other fact follows by definition? Compare the circumferences of the two bases; their areas. Compare the total areas of the two cones; their volumes.

Ex. 1429. If the lateral area of a right circular cone is $1\frac{9}{16}$ times that of a similar cone, what is the ratio of their volumes? of their altitudes?

Ex. 1430. Through a given cone X two planes are passed parallel to the base; let Y denote the cone cut off by the upper plane, and Z the entire cone cut off by the lower plane. Prove that Y and Z are to each other as the cubes of the distances of the planes from the vertex of the given cone X.

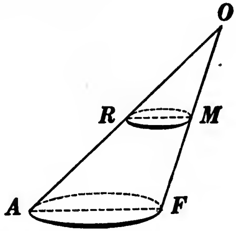
Hint. See Ex. 1381.

Ex. 1431. Show that § 897 is a special case of Ex. 1430.

Ex. 1432. The lateral area of a cone of revolution is 144 square inches and the total area 240 square inches. Find the volume.

PROPOSITION XIX. THEOREM

898. *The volume of a frustum of a cone is equal to one third the product of its altitude and the sum of its lower base, its upper base, and the mean proportional between its two bases.*



Given frustum AM, of cone $O\text{–}AF$, with its volume denoted by V, its lower base by B, its upper base by b, and its altitude by H.

To prove $V = \frac{1}{3}H(B + b + \sqrt{B \cdot b})$.

The proof is left as an exercise for the student.

HINT. In the proof of § 815, change "pyramid" to "cone."

899. Cor. *If V denotes the volume, H the altitude, and R and r the radii of the bases of a frustum of a cone,*

$$V = \frac{1}{3}\pi H(R^2 + r^2 + R \cdot r).$$

Ex. 1433. Make a frustum of a right circular cone as indicated in Ex. 1401, and of the same dimensions. Find its volume.

Ex. 1434. A tin pail is in the form of a frustum of a cone; the diameter of its upper base is 12 inches, of its lower base 10 inches. How high must the pail be to hold $2\frac{1}{2}$ gallons of water? (See Ex. 1422.)

Ex. 1435. A cone 6 feet high is cut by a plane parallel to the base and 4 feet from the vertex; the volume of the frustum formed is 456 cubic inches. Find the volume of the entire cone.

Ex. 1436. Find the ratio of the base to the lateral area of a right circular cone whose altitude is equal to the diameter of its base.

MISCELLANEOUS EXERCISES

Ex. 1437. The base of a cylinder is inscribed in a face of a cube whose edge is 10 inches. Find the altitude of the cylinder if its volume is equal to the volume of the cube.

Ex. 1438. A block of marble in the form of a regular prism is 10 feet long and 2 feet 6 inches square at the base. Find the volume of the largest cylindrical pillar that can be cut from it.

Ex. 1439. The Winchester bushel, formerly used in England, was the volume of a right circular cylinder $18\frac{1}{2}$ inches in internal diameter and 8 inches in depth. Is this the same volume as the bushel used in the United States (2150.42 cubic inches)?

Ex. 1440. To determine the volume of an irregular body, it was immersed in a vessel containing water. The vessel was in the form of a right circular cylinder the radius of whose base was 8 inches. On placing the body in the cylinder, the surface of the water was raised $10\frac{1}{2}$ inches. Find the volume of the irregular solid.

Ex. 1441. In draining a certain pond a 4-inch tiling (*i.e.* a tiling whose inside diameter was 4 inches) was used. In draining another pond, supposed to contain half as much water, a 2-inch tiling was laid. It could not drain the pond. What was the error made?

Ex. 1442. A grain elevator in the form of a frustum of a right circular cone is 25 feet high; the radii of its bases are 10 feet and 5 feet, respectively; how many bushels of wheat will it hold, counting $1\frac{1}{4}$ cubic feet to a bushel?

Ex. 1443. The altitude of a cone with circular base is 16 inches. At what distance from the vertex must a plane be passed parallel to the base to cut the cone into two equivalent parts?

Ex. 1444. Two sides of a triangle including an angle of 120° are 10 and 20, respectively. Find the volume of the solid generated by revolving the triangle about side 10 as an axis.

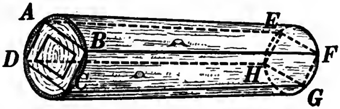
Ex. 1445. Find the volume of the solid generated by revolving the triangle of Ex. 1444 about side 20 as an axis.

Ex. 1446. Find the volume of the solid generated by revolving the triangle of Ex. 1444 about its longest side as an axis.

Ex. 1447. The slant height of a right circular cone is 20 inches, and the circumference of its base 4π inches. A plane parallel to the base cuts off a cone whose slant height is 8 inches. Find the lateral area and the volume of the frustum remaining.

Ex. 1448. A cone has an altitude of 12.5 feet and a base whose radius is 8.16 feet; the base of a cylinder having the same volume as the cone has a radius of 6.25 feet. Find the altitude of the cylinder.

Ex. 1449. A log 20 feet long is 3 feet in diameter at the top end and 4 feet in diameter at the butt end.



(*a*) How many cubic feet of wood does the log contain?

(*b*) How many cubic feet are there in the largest piece of square timber that can be cut from the log?

(*c*) How many cubic feet in the largest piece of square timber the same size throughout its whole length?

(*d*) How many board feet does the piece of timber in (*c*) contain, a board foot being equivalent to a board 1 foot square and 1 inch thick?

HINT. In (*b*) the larger end is square $ABCD$. What is the smaller end? In (*c*) one end is square $EFGH$. What is the other end?

Ex. 1450. The base of a cone has a radius of 16 inches. A section of the cone through the vertex, through the center of the base, and perpendicular to the base, is a triangle two of whose sides are 20 inches and 24 inches, respectively. Find the volume of the cone.

Ex. 1451. The hypotenuse of a right triangle is 10 inches and one side 8 inches; find the area of the surface generated by revolving the triangle about its hypotenuse as an axis.

Ex. 1452. A tin pail in the form of a frustum of a right circular cone is 8 inches deep; the diameters of its bases are $8\frac{1}{2}$ inches and $10\frac{1}{2}$ inches, respectively. How many gallons of water will it hold? (One liquid gallon contains 231 cubic inches.)

Ex. 1453. The altitude of a cone is 12 inches. At what distances from the vertex must planes be passed parallel to the base to divide the cone into four equivalent parts?

HINT. See Ex. 1430.

Ex. 1454. Find the volume of the solid generated by an equilateral triangle, whose side is a, revolving about one of its sides as an axis.

Ex. 1455. Regular hexagonal prisms are inscribed in and circumscribed about a right circular cylinder. Find (*a*) the ratio of the lateral areas of the three solids; (*b*) the ratio of their total areas; (*c*) the ratio of their volumes.

Ex. 1456. How many miles of platinum wire $\frac{1}{20}$ of an inch in diameter can be made from 1 cubic foot of platinum?

Ex. 1457. A tank in the form of a right circular cylinder is 5 feet long and the radius of its base is 8 inches. If placed so that its axis is horizontal and filled with gasoline to a depth of 12 inches, how many gallons of gasoline will it contain?

HINT. See Ex. 1024.

Ex. 1458. Find the weight in pounds of an iron pipe 10 feet long, if the iron is $\frac{1}{2}$ inch thick and the outer diameter of the pipe is 4 inches. (1 cubic foot of bar iron weighs 7780 ounces.)

Ex. 1459. In a certain right circular cone whose altitude and radius of base are equal, the total surface and the volume have the same numerical value. Find the volume of the cone.

Ex. 1460. Two cones of revolution lie on opposite sides of a common base. Their slant heights are 12 and 5, respectively, and the sum of their altitudes is 13. Find the radius of the common base.

Ex. 1461. The radii of the lower and upper bases of a frustum of a right circular cone are R and R', respectively. Show that the area of a section midway between them is $\frac{\pi(R + R')^2}{4}$.

Ex. 1462. A plane parallel to the base of a right circular cone leaves three fourths of the cone's volume. How far from the vertex is this plane? How far from the vertex is the plane if it cuts off half the volume? Answer the same questions for a cylinder.

Ex. 1463. Is every cone cut from a right circular cone by a plane parallel to its base necessarily similar to the original cone? why? How is it with a cylinder? why?

Ex. 1464. Water is carried from a spring to a house, a distance of $\frac{1}{2}$ mile, in a cylindrical pipe whose inside diameter is 2 inches. How many gallons of water are contained in the pipe?

Ex. 1465. A square whose side is 6 inches is revolved about one of its diagonals as an axis. Find the surface and the volume of the solid generated. Can you find the volume of the solid generated by revolving a cube about one of its diagonals as an axis?

HINT. Make a cube of convenient size from pasteboard, pass a hatpin through two diagonally opposite vertices, and revolve the cube *rapidly*.

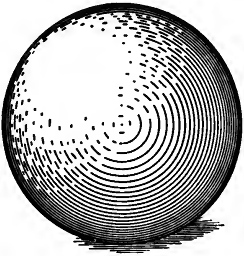
Ex. 1466. Given V the volume, and R the radius of the base, of a right circular cylinder. Find the lateral area and total area.

Ex. 1467. Given the total area T, and the altitude H, of a right circular cylinder. Find the volume.

BOOK IX

THE SPHERE

900. Def. A **sphere** is a solid closed figure whose boundary is a curved surface such that all straight lines to it from a fixed point within are equal.



901. Defs. The fixed point within the sphere is called its **center**; a straight line joining the center to any point on the surface is a **radius**; a straight line passing through the center and having its extremities on the surface is a **diameter**.

902. From the above definitions and from the definition of equal figures, § 18, it follows that:

(*a*) *All radii of the same sphere, or of equal spheres, are equal.*

(*b*) *All diameters of the same sphere, or of equal spheres, are equal.*

(*c*) *Spheres having equal radii, or equal diameters, are equal.*

(*d*) *A sphere may be generated by the revolution of a circle about a diameter as an axis.*

Ex. 1468. Find the locus of all points that are 3 inches from the surface of a sphere whose radius is 7 inches.

Ex. 1469. The three edges of a trihedral angle pierce the surface of a sphere. Find the locus of all points of the sphere that are:

(*a*) Equidistant from the three edges of the trihedral angle.

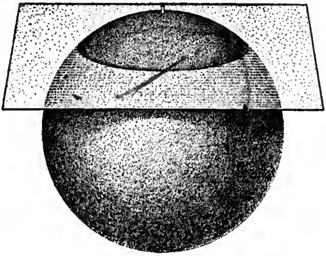
(*b*) Equidistant from the three faces of the trihedral angle.

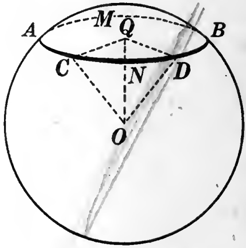
Ex. 1470. Find a point in a plane equidistant from three given points in space.

Ex. 1471. Find the locus of all points in space equidistant from the three sides of a given triangle.

Proposition I. Theorem

903. *Every section of a sphere made by a plane is a circle.*

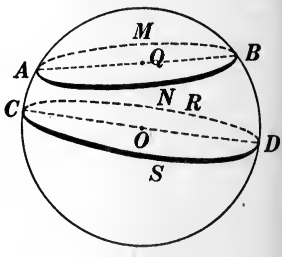




Given $AMBN$ a section of sphere O made by a plane.

To prove section $AMBN$ a $\odot$.

Argument	Reasons
1. From O draw $OQ \perp$ section $AMBN$.	1. § 639.
2. Join Q to C and D, any two points on the perimeter of section $AMBN$. Draw OC and OD.	2. § 54, 15.
3. In rt. $\triangle$ OQC and OQD, $OQ = OQ$.	3. By iden.
4. $OC = OD$.	4. § 902, *a*.
5. $\therefore \triangle OQC = \triangle OQD$.	5. § 211.
6. $\therefore QC = QD$; *i.e.* any two points on the perimeter of section $AMBN$ are equidistant from Q.	6. § 110.
7. $\therefore$ section $AMBN$ is a $\odot$. Q.E.D.	7. § 276.



904. Def. A **great circle of a sphere** is a section made by a plane which passes through the center of the sphere, as $\odot$ $CRDS$.

905. Def. A **small circle of a sphere** is a section made by a plane which does not pass through the center of the sphere, as $\odot$ $AMBN$.

906. Def. The **axis of a circle of a sphere** is the diameter of the sphere which is perpendicular to the plane of the circle.

907. Def. The **poles of a circle of a sphere** are the extremities of the axis of the circle.

Ex. 1472. Considering the earth as a sphere, what kind of circles are the parallels of latitude? the equator? the meridian circles? What is the axis of the equator? of the parallels of latitude? What are the poles of the equator? of the parallels of latitude?

Ex. 1473. The radius of a sphere is 17 inches. Find the area of a section made by a plane 8 inches from the center.

Ex. 1474. The area of a section of a sphere 45 centimeters from the center is 784π square centimeters. Find the radius of the sphere.

Ex. 1475. The area of a section of a sphere 7 inches from the center is 576π. Find the area of a section 8 inches from the center.

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908. The following are some of the properties of a sphere; the student should prove the correctness of each:

(*a*) *In equal spheres, or in the same sphere, if two sections are equal, they are equally distant from the center, and conversely.*

HINT. Compare with § 307.

(*b*) *In equal spheres, or in the same sphere, if two sections are unequal, the greater section is at the less distance from the center, and conversely.* (HINT. See §§ 308, 310.)

(*c*) *In equal spheres, or in the same sphere, all great circles are equal.* (HINT. See § 279, c.)

(*d*) *The axis of a small circle of a sphere passes through the center of the circle, and conversely.*

(*e*) *Any two great circles of a sphere bisect each other.*

(*f*) *Every great circle of a sphere bisects the surface and the sphere.*

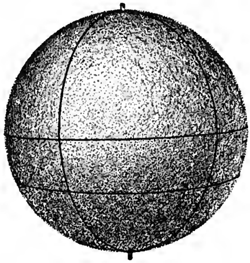
(*g*) *Through two points on the surface of a sphere, not the extremities of a diameter, there exists one and only one great circle.*

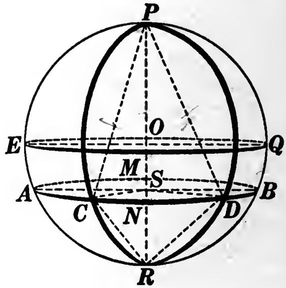
(*h*) *Through three points on the surface of a sphere there exists one and only one circle.*

909. Def. The **distance between two points on the surface of a sphere** is the length of the minor arc of the great circle joining them.

PROPOSITION II. THEOREM

910. *All points on the circumference of a circle of a sphere are equidistant from either pole of the circle.*





Given C and D any two points on the circumference, and P and R the poles, of $\odot$ $AMBN$.

To prove arc $PC =$ arc PD and arc $RC =$ arc RD.

The proof is left as an exercise for the student.

HINT. Apply § 298.

911. Def. The **polar distance of a circle of a sphere** is the distance between any point on its circumference and the nearer pole of the circle.

912. Cor. I. *The polar distance of a great circle is a quadrant.*

913. Cor. II. *In equal spheres, or in the same sphere, the polar distances of equal circles are equal.*

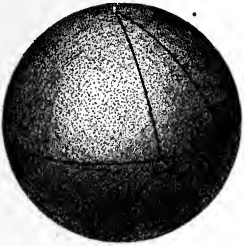
Ex. 1476. What is the locus of all points on the surface of the earth at a quadrant's distance from the north pole? from the south pole? from the equator? from a point P on the equator? at a distance of $23\frac{1}{2}°$ from the south pole? $23\frac{1}{2}°$ from the equator? 180° from the north pole?

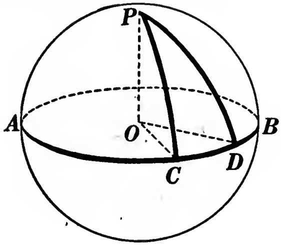
Ex. 1477. Considering the earth as a sphere with a radius of 4000 miles, calculate in miles the polar distance of: (*a*) the Arctic Circle; (*b*) the Tropic of Cancer; (*c*) the equator.

Ex. 1478. State a postulate for the construction of a circle on the surface of a sphere corresponding to § 122, the postulate for the construction of a circle in a plane.

PROPOSITION III. THEOREM

914. *A point on the surface of a sphere at the distance of a quadrant from each of two other points (not the extremities of the same diameter) on the surface, is the pole of the great circle passing through these two points.*





Given PC and PD quadrants of great ⓢ of sphere O, and $ACDB$ a great ⊙ passing through points C and D.

To prove P the pole of great ⊙ $ACDB$.

ARGUMENT	REASONS
1. Draw OC, OD, and OP.	1. § 54, 15.
2. $\overset{\frown}{PC} = 90°$ and $\overset{\frown}{PD} = 90°$.	2. By hyp
3. $\therefore$ ∠s POC and POD are rt. ∠s; *i.e.* $OP \perp OC$ and OD	3. § 358.
4. $\therefore OP \perp$ the plane of ⊙ $ACDB$.	4. § 622.
5. $\therefore OP$ is the axis of ⊙ $ACDB$.	5. § 906.
6 $\therefore P$ is the pole of great ⊙ $ACDB$. Q.E.D.	6. § 907.

Ex. 1479. Assuming the chord of a quadrant of a great circle of a sphere to be given, construct with compasses an arc of a great circle through two given points on the surface of the sphere.

Ex. 1480. If the planes of two great circles are perpendicular to each other, each passes through the poles of the other.

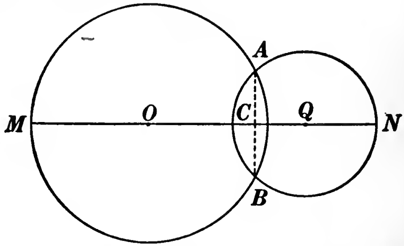
Ex. 1481. Find the locus of all points in space equidistant from two given points and at a given distance d from a third given point.

Ex. 1482. Find the locus of all points in space at a distance d from a given point and at a distance m from a given plane.

Ex. 1483. Find the locus of all points in space equidistant from two given points and also equidistant from two given parallel lines.

Proposition IV. Theorem

915. *The intersection of two spherical surfaces is the circumference of a circle.*



Given two spherical surfaces generated by intersecting circumferences O and Q revolving about line MN as an axis.

To prove the intersection of the two spherical surfaces the circumference of a ⊙.

Outline of Proof

1. Show that $MN \perp AB$ at its mid-point C (§ 328). bisects common cord
2. Show that AC, revolving about axis MN, generates a plane.
3. Show that A generates the circumference of a ⊙.
4. The locus of A is the intersection of what (§ 614)?
5. ∴ the intersection of the two spherical surfaces is the circumference of a ⊙. Q.E.D.

Ex. 1484. Find the locus of all points in space 6 inches from a given point P and 10 inches from another given point Q.

Ex. 1485. The radii of two intersecting spheres are 12 inches and 16 inches, respectively. The line joining their centers is 24 inches. Find the circumference and area of their circle of intersection.

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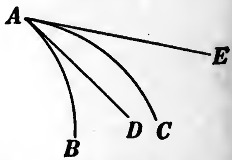
916. Def. An **angle formed by two intersecting arcs** of circles is the angle formed by tangents to the two arcs at their point of intersection; thus the ∠ formed by arcs AB and AC is plane ∠ DAE.

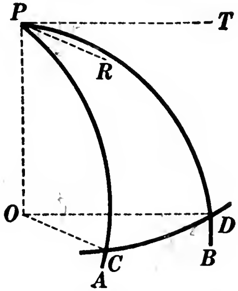


917. Def. A **spherical angle** is an angle formed by two intersecting arcs of great circles of a sphere.*

* A different meaning is sometimes attached to the expression "spherical angle" in advanced mathematics.

Proposition V. Theorem

918. *A spherical angle is measured by the arc of a great circle having the vertex of the angle as a pole and intercepted by the sides of the angle, prolonged if necessary.*



Given spherical $\angle APB$, with CD an arc of a great $\odot$ whose pole is P and which is intercepted by sides PA and PB of $\angle APB$.

To prove that $\angle APB \overset{\infty}{=} \overset{\frown}{CD}$.

Outline of Proof

1. Draw radii OP, OC, and OD.
2. From P draw PR tangent to $\overset{\frown}{PA}$ and PT tangent to $\overset{\frown}{PB}$.
3. Prove OC and OD each $\perp OP$.
4. Prove $OC \parallel PR$, $OD \parallel PT$, and hence $\angle COD = \angle RPT$.
5. But $\angle COD \overset{\infty}{=} \overset{\frown}{CD}$; $\therefore \angle RPT$, *i.e.* $\angle APB \overset{\infty}{=} \overset{\frown}{CD}$. Q.E.D.

919. Cor. I. *A spherical angle is equal to the plane angle of the dihedral angle formed by the planes of the sides of the angle.*

920. Cor. II. *The sum of all the spherical angles about a point on the surface of a sphere equals four right angles.*

Ex. 1486. By comparison with the definitions of the corresponding terms in plane geometry, frame exact definitions of the following classes of spherical angles: acute, right, obtuse, adjacent, complementary, supplementary, vertical.

Ex. 1487. Any two vertical spherical angles are equal.

Ex. 1488. If one great circle passes through the pole of another great circle, the circles are perpendicular to each other.

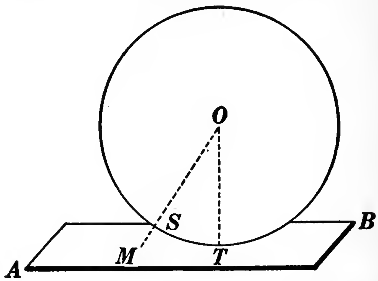
LINES AND PLANES TANGENT TO A SPHERE

921. Def. A **straight line** or a **plane** is **tangent to a sphere** if, however far extended, it meets the sphere in one and only one point.

922. Def. **Two spheres** are **tangent to each other** if they have one and only one point in common. They are tangent **internally** if one sphere lies within the other, and **externally** if neither sphere lies within the other.

Proposition VI. Theorem

923. *A plane tangent to a sphere is perpendicular to the radius drawn to the point of tangency.*



Given plane AB tangent to sphere O at T, and OT a radius drawn to the point of tangency.

To prove plane $AB \perp OT$.

The proof is left as an exercise for the student.

924. Question. What changes are necessary in the proof of § 313 to make it the proof of § 923?

925. Cor. I. (Converse of Prop. VI). *A plane perpendicular to a radius of a sphere at its outer extremity is tangent to the sphere.*

Hint. See § 314.

Ex. 1489. A straight line tangent to a sphere is perpendicular to the radius drawn to the point of tangency.

Ex. 1490. State and prove the converse of Ex. 1489.

Ex. 1491. Two lines tangent to a sphere at the same point determine a plane tangent to the sphere at that point.

Ex. 1492. Given a point P on the surface of sphere O. Explain how to construct: (a) a line tangent to sphere O at P; (b) a plane tangent to sphere O at P.

Ex. 1493. Given a point R outside of sphere Q. Explain how to construct: (a) a line through R tangent to sphere Q; (b) a plane through R tangent to sphere Q.

Hint. Compare with § 373.

Ex. 1494. Two planes tangent to a sphere at the extremities of a diameter are parallel.

Ex. 1495. If the straight line joining the centers of two spheres is equal to the sum of their radii, the spheres are tangent to each other.

Hint. Show that the radius of the ⊙ of intersection of the two spheres (§ 915) is zero.

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926. Def. A **polyhedron** is **circumscribed about a sphere** if each face of the polyhedron is tangent to the sphere.

927. Def. If a polyhedron is circumscribed about a sphere, the **sphere** is said to be **inscribed in the polyhedron.**

928. Def. A **polyhedron** is **inscribed in a sphere** if all its vertices are on the surface of the sphere.

929. Def. If a polyhedron is inscribed in a sphere, the **sphere** is said to be **circumscribed about the polyhedron.**

Ex. 1496. Find the edge of a cube inscribed in a sphere whose radius is 10 inches.

Ex. 1497. Find the volume of a cube: (a) circumscribed about a sphere whose radius is 8 inches; (b) inscribed in a sphere whose radius is 8 inches.

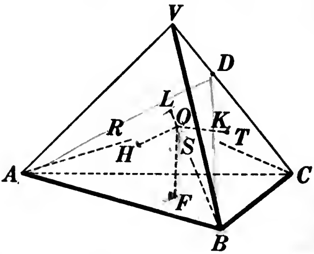
Ex. 1498. A right circular cylinder whose altitude is 8 inches is inscribed in a sphere whose radius is 6 inches. Find the volume of the cylinder.

Ex. 1499. A right circular cone, the radius of whose base is 8 inches, is inscribed in a sphere with radius 12 inches. Find the volume of the cone.

Ex. 1500. Find the volume of a right circular cone circumscribed about a regular tetrahedron whose edge is a.

Proposition VII. Problem

930. *To inscribe a sphere in a given tetrahedron.*



Given tetrahedron *V–ABC*.

To inscribe a sphere in tetrahedron *V–ABC*.

I. Construction

1. Construct planes *RABS*, *SBCT*, and *TCAR* bisecting dihedral ∠s whose edges are *AB*, *BC*, and *CA*, respectively. § 691.

2. From *O*, the point of intersection of the three planes, construct $OF \perp$ plane *ABC*. § 637.

3. The sphere constructed with *O* as center and *OF* as radius will be inscribed in tetrahedron *V–ABC*.

II. Proof

Argument	Reasons
1. Plane *RABS*, the bisector of dihedral ∠ *AB*, lies between planes *ABV* and *ABC*; *i.e.* it intersects edge *VC* in some point as *D*.	1. By cons.
2. ∴ plane *RABS* intersects plane *BCV* in line *BD* and plane *ACV* in line *AD*.	2. § 616.
3. Plane *SBCT* lies between planes *BCV* and *ABC*; *i.e.* it intersects plane *RASB* in a line through *B* between *BA* and *BD*, as *BS*.	3. By cons.
4. Similarly plane *TCAR* intersects plane *RABS* in a line through *A* between	4. By steps similar to 1–3.

Argument	Reasons
AB and AD, as AR; and plane $SBCT$ intersects plane $TCAR$ in a line through C as CT.	
5. But AR and BS pass through the interior of $\triangle ABD$.	5. Args. 3 & 4.
6. $\therefore$ AR and BS intersect in some point as O, within $\triangle ABD$.	6. § 194.
7. $\therefore$ AR, BS, and CT are concurrent in point O.	7. § 617, I.
8. From O draw OH, OK, and $OL \perp$ planes VAB, VBC, and VCA, respectively.	8. § 639.
9. $\because$ O is in plane OAB, $OF = OH$.	9. § 688.
10. $\because$ O is in plane OBC, $OF = OK$.	10. § 688.
11. $\because$ O is in plane OCA, $OF = OL$.	11. § 688.
12. $\therefore$ $OF = OH = OK = OL$.	12. § 54, 1.
13. $\therefore$ each of the four faces of the tetrahedron is tangent to sphere O.	13. § 925.
14. $\therefore$ sphere O is inscribed in tetrahedron $V\text{–}ABC$. Q.E.D.	14. §§ 926, 927.

III. Discussion

The discussion is left as an exercise for the student.

Ex. 1501. The six planes bisecting the dihedral angles of a tetrahedron meet in a point which is equidistant from the four faces of the tetrahedron.

Hint. Compare with § 258.

Ex. 1502. Inscribe a sphere in a given cube.

Ex. 1503. The volume of any tetrahedron is equal to the product of its surface and one third the radius of the inscribed sphere.

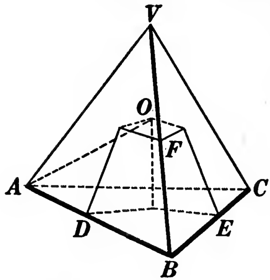
Hint. See §§ 491 and 492.

Ex. 1504. Find a point within a triangular pyramid such that the planes determined by the lines joining this point to the vertices shall divide the pyramid into four equivalent parts.

Hint. Compare with Ex. 1094.

Proposition VIII. Problem

931. *To circumscribe a sphere about a given tetrahedron.*



Given tetrahedron $V\text{–}ABC$.

To circumscribe a sphere about tetrahedron $V\text{–}ABC$.

I. Construction

1. Through D, the mid-point of AB, construct plane $DO \perp AB$; through E, the mid-point of BC, construct plane $EO \perp BC$; and through F, the mid-point of VB, construct plane $FO \perp VB$.

2. Join O, the point of intersection of the three planes, to A.

3. The sphere constructed with O as center and OA as radius will be circumscribed about tetrahedron $V\text{–}ABC$.

II. Outline of Proof

1. Prove that the three planes OD, OE, and OF intersect each other in three lines.

2. Prove that these three lines of intersection meet in a point, as O.

3. Prove that $OA = OB = OC = OV$.

4. $\therefore$ sphere O is circumscribed about tetrahedron $V\text{–}ABC$.

Q.E.D.

932. Cor. *Four points not in the same plane determine a sphere.*

933. Questions. Are the methods used in §§ 930 and 931 similar to those used in §§ 321 and 323? In § 930 could the lines forming the edges of the dihedral angles bisected be three lines meeting in one vertex? In § 931 could the three edges bisected be three lines lying in the same face?

Ex. 1505. The six planes perpendicular to the edges of a tetrahedron at their mid-points meet in a point which is equidistant from the four vertices of the tetrahedron.

Ex. 1506. The four lines perpendicular to the faces of a tetrahedron, and erected at their centers, meet in a point which is equidistant from the four vertices of the tetrahedron.

Ex. 1507. Circumscribe a sphere about a given cube.

Ex. 1508. Circumscribe a sphere about a given rectangular parallelopiped. Can a sphere be inscribed in any rectangular parallelopiped? Explain.

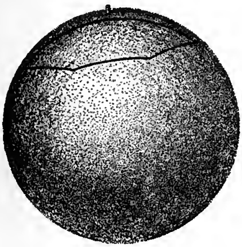
Ex. 1509. Find a point equidistant from four points in space not all in the same plane.

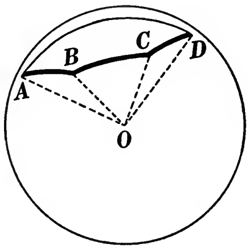
SPHERICAL POLYGONS

934. Def. A line on the surface of a sphere is said to be **closed** if it separates a portion of the surface from the remaining portion.

935. Def. A **closed figure** on the surface of a sphere is a figure composed of a portion of the surface of the sphere and its bounding line or lines.

936. Defs. A **spherical polygon** is a closed figure on the surface of a sphere whose boundary is composed of three or more arcs of great circles, as *ABCD*.





The bounding arcs are called the **sides** of the polygon, the points of intersection of the arcs are the **vertices** of the polygon, and the spherical angles formed by the sides are the **angles** of the polygon.

937. Def. A **diagonal of a spherical polygon** is an arc of a great circle joining any two non-adjacent vertices.

938. Def. A **spherical triangle** is a spherical polygon having three sides.

Ex. 1510. By comparison with the definitions of the corresponding terms in plane geometry, frame exact definitions of the following classes of spherical triangles: scalene, isosceles, equilateral, acute, right, obtuse, and equiangular.

Ex. 1511. With a given arc as one side, construct an equilateral spherical triangle.

HINT. Compare the cons., step by step, with § 124.

Ex. 1512. With three given arcs as sides, construct a scalene spherical triangle.

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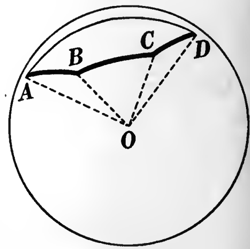
939. Since each side of a spherical polygon is an arc of a great circle (§ 936), the planes of these arcs meet at the center of the sphere and form at that point a polyhedral angle, as polyhedral $\angle$ O–$ABCD$.



This polyhedral angle and the spherical polygon are very closely related. The following are some of the more important relations; the student should prove the correctness of each:

940. (*a*) *The sides of a spherical polygon have the same measures as the corresponding face angles of the polyhedral angle.*

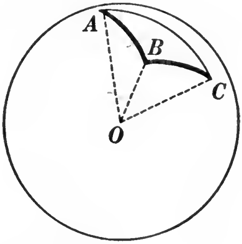
(*b*) *The angles of a spherical polygon have the same measures as the corresponding dihedral angles of the polyhedral angle.*

Thus, sides AB, BC, etc., of spherical polygon $ABCD$ have the same measures as face $\angle$s AOB, BOC, etc., of polyhedral $\angle$ O–$ABCD$; and spherical $\angle$s ABC, BCD, etc., have the same measures as the dihedral $\angle$s whose edges are OB, OC, etc.

These relations make it possible to establish certain properties of spherical polygons from the corresponding known properties of the polyhedral angle, as in §§ 941 and 942.

Proposition IX. Theorem

941. *The sum of any two sides of a spherical triangle is greater than the third side.*



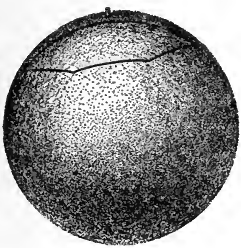
Given spherical $\triangle ABC$.

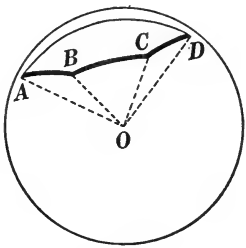
To prove $\overset{\frown}{AB} + \overset{\frown}{BC} > \overset{\frown}{CA}$.

Argument	Reasons
1. $\angle AOB + \angle BOC > \angle COA$.	1. § 710.
2. $\angle AOB \propto \overset{\frown}{AB}$, $\angle BOC \propto \overset{\frown}{BC}$, $\angle COA \propto \overset{\frown}{CA}$.	2. § 940, *a*.
3. $\therefore \overset{\frown}{AB} + \overset{\frown}{BC} > \overset{\frown}{CA}$. Q.E.D.	3. § 362, *b*.

Proposition X. Theorem

942. *The sum of the sides of any spherical polygon is less than 360°.*





Given spherical polygon $ABC \cdots$ with n sides.

To prove $\overset{\frown}{AB} + \overset{\frown}{BC} + \cdots < 360°$.

Hint. See §§ 712 and 940, *a*.

Ex. 1513. In spherical triangle ABC, arc $AB = 40°$ and arc $BC = 80°$. Between what limits must arc CA lie?

Ex. 1514. Any side of a spherical polygon is less than the sum of the remaining sides.

Ex. 1515. If arc AB is the perpendicular bisector of arc CD, every point on the surface of the sphere and not in arc AB is unequally distant from C and D.

Ex. 1516. In a spherical quadrilateral, between what limits must the fourth side lie if three sides are 60°, 70°, and 80°? if three sides are 40°, 50°, and 70°?

Ex. 1517. Any side of a spherical polygon is less than 180°.

HINT. See Ex. 1514.

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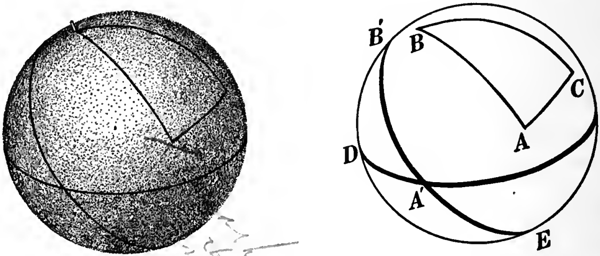
943. If, with A, B, and C the vertices of any spherical triangle as poles, three great circles are constructed, as $B'C'ED$,



$C'A'D$, and $EA'B'$, the surface of the sphere will be divided into eight spherical triangles, four of which are seen on the hemisphere represented in the figure. Of these eight spherical triangles, $A'B'C'$ is the one and only one that is so situated that A and A' lie on the same side of BC, B and B' on the same side of AC, and C and C' on the same side of AB. This particular triangle $A'B'C'$ is called the **polar triangle** of triangle ABC.

944. Questions. In the figure above, $\triangle A'B'C'$, the polar of $\triangle ABC$, is entirely outside of $\triangle ABC$. Can the two $\triangle$s be so constructed that:

(a) $A'B'C'$ is entirely within ABC?

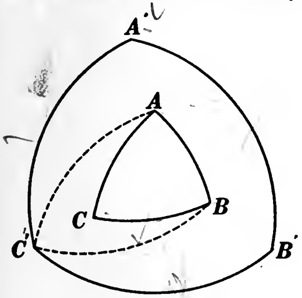
(b) $A'B'C'$ is partly outside of and partly within ABC?

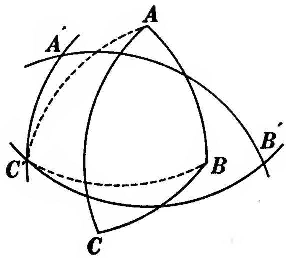
Ex. 1518. What is the polar triangle of a spherical triangle all of whose sides are quadrants?

Know

Proposition XI. Theorem

945. *If one spherical triangle is the polar of another, then the second is the polar of the first.*





Given $\triangle A'B'C'$ the polar of $\triangle ABC$.

To prove $\triangle ABC$ the polar of $\triangle A'B'C'$.

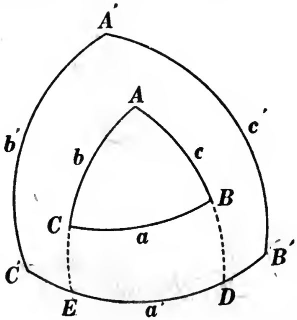
Argument	Reasons
1. A is the pole of $\widehat{B'C'}$; *i.e.* $\widehat{AC'}$ is a quadrant.	1. § 912.
2. B is the pole of $\widehat{A'C'}$; *i.e.* $\widehat{BC'}$ is a quadrant.	2. § 912.
3. $\therefore$ C' is the pole of $\widehat{AB}$.	3. § 914.
4. Likewise B' is the pole of $\widehat{AC}$, and A' is the pole of $\widehat{BC}$.	4. By steps similar to 1–3.
5. $\therefore$ $\triangle ABC$ is the polar of $\triangle A'B'C'$. Q.E.D.	5. § 943.

946. Historical Note. The properties of polar triangles were discovered about 1626 A.D. by Albert Girard, a Dutch mathematician, born in Lorraine about 1595. They were also discovered independently and about the same time by Snell, an "infant prodigy," who at the age of twelve was familiar with the standard mathematical works of that time and who is remembered as the discoverer of the well-known law of refraction of light.

Ex. 1519. Determine the polar triangle of a spherical triangle having two of its sides quadrants and the third side equal to $70°$; $110°$; $(90-a)°$; $(90+a)°$.

Proposition XII. Theorem

947. *In two polar triangles each angle of one and that side of the other of which its vertex is the pole are together equal, numerically, to 180°.*



Given polar △ ABC and $A'B'C'$, with sides denoted by a, b, c, and a', b', c', respectively.

To prove: (*a*) $\angle A + a' = 180°$, $\angle B + b' = 180°$, $\angle C + c' = 180°$;
(*b*) $\angle A' + a = 180°$, $\angle B' + b = 180°$, $\angle C' + c = 180°$.

(*a*) Argument Only

1. Let arcs AB and AC (prolonged if necessary) intersect arc $B'C'$ at D and E, respectively; then $C'D = 90°$ and $EB' = 90°$.
2. $\therefore C'D + EB' = 180°$.
3. $\therefore C'E + ED + ED + DB' = 180°$; *i.e.* $ED + a' = 180°$.
4. But ED is the measure of $\angle A$.
5. $\therefore \angle A + a' = 180°$.
6. Likewise $\angle B + b' = 180°$, and $\angle C + c' = 180°$. Q.E.D.

(*b*) The proof of (*b*) is left as an exercise for the student.

Hint. Let BC prolonged meet $A'B'$ at H and $A'C'$ at K.

948. Question. In the history of mathematics, why are polar triangles frequently spoken of as **supplemental triangles**?

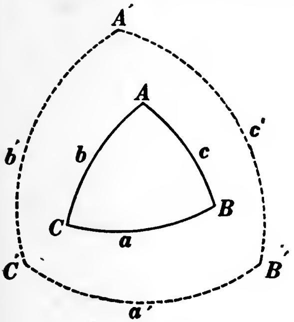
Ex. 1520. The angles of a spherical triangle are 75°, 85°, and 145°. Find the sides of its polar triangle.

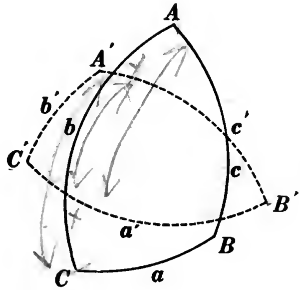
Ex. 1521. If a spherical triangle is equilateral, its polar triangle is equiangular; and conversely.

Know

Proposition XIII. Theorem

949. *The sum of the angles of a spherical triangle is greater than 180° and less than 540°.*





Given spherical $\triangle\, ABC$ with sides denoted by a, b, and c.

To prove $\angle A + \angle B + \angle C > 180°$ and $< 540°$.

Argument	Reasons
1. Let $\triangle\, A'B'C'$, with sides denoted by a', b', and c', be the polar of $\triangle\, ABC$.	1. § 943.
2. Then $\angle A + a' = 180°$, $\angle B + b' = 180°$, $\angle C + c' = 180°$.	2. § 947.
3. $\therefore \angle A + \angle B + \angle C + (a' + b' + c') = 540°$.	3. § 54, 2.
4. But $a' + b' + c' < 360°$.	4. § 942.
5. $\therefore \angle A + \angle B + \angle C > 180°$.	5. § 54, 6.
6. Again, $a' + b' + c' > 0°$.	6. § 938.
7. $\therefore \angle A + \angle B + \angle C < 540°$. Q.E.D.	7. § 54, 6.

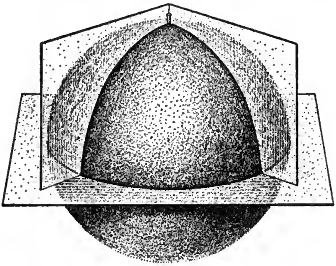
950. Cor. *In a spherical triangle there can be one, two, or three right angles; there can be one, two, or three obtuse angles.*

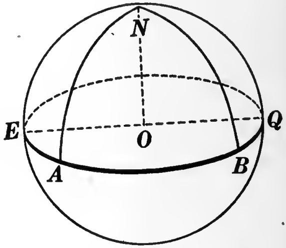
951. Note. Throughout the Solid Geometry the student's attention has constantly been called to the relations between definitions and theorems of solid geometry and the corresponding definitions and theorems of plane geometry. In the remaining portion of the geometry of the sphere there will likewise be many of these comparisons, but here the student must be particularly on his guard for contrasts as well as comparisons. For ex-

ample, while the sum of the angles of a plane triangle is equal to exactly 180°, he has learned (§ 949) that the sum of the angles of a spherical triangle may be any number from 180° to 540°; while a plane triangle can have but one right or one obtuse angle, a spherical triangle may have one, two, or three right angles or one, two, or three obtuse angles (§ 950).

If the student will recall that, considering the earth as a sphere, the north and south poles of the earth are the poles of the equator, and that all meridian circles are great circles perpendicular to the equator, it will make his thinking about spherical triangles more definite.

952. Def. A spherical triangle containing two right angles is called a **birectangular spherical triangle.**





953. Def. A spherical triangle having all of its angles right angles is called a **trirectangular spherical triangle.**

Thus two meridians, as *NA* and *NB*, making at the north pole an acute or an obtuse ∠, form with the equator a birectangular spherical △. If the ∠ between *NA* and *NB* is made a rt. ∠, △ *ANB* becomes a trirectangular spherical △.

Ex. 1522. What kind of arcs are *NA* and *NB*? Then what arc measures spherical angle *ANB*? Are two sides of any birectangular spherical triangle quadrants? What is each side of a trirectangular spherical triangle?

Ex. 1523. If two sides of a spherical triangle are quadrants, the triangle is birectangular. (HINT. Apply § 947.)

Ex. 1524. What is the polar triangle of a trirectangular spherical triangle?

Ex. 1525. An exterior angle of a spherical triangle is less than the sum of the two remote interior angles. Compare this exercise with § 215. Make this new fact clear by applying it to a birectangular spherical triangle whose third angle is: (*a*) acute; (*b*) right; (*c*) obtuse.

Proposition XIV. Theorem

954. *In equal spheres, or in the same sphere, two spherical triangles are equal:*

I. *If a side and the two adjacent angles of one are equal respectively to a side and the two adjacent angles of the other;*

II. *If two sides and the included angle of one are equal respectively to two sides and the included angle of the other;*

III. *If the three sides of one are equal respectively to the three sides of the other:*

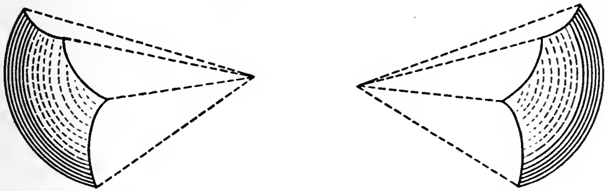
provided the equal parts are arranged in the same order.

The proofs are left as exercises for the student.

Hint. In each of the above cases prove the corresponding trihedral ∠s equal (§§ 702, 704); and thus show that the spherical △s are equal.

955. Questions. Compare Prop. XIV, I and II, with § 702, I and II, and with §§ 105 and 107. Could the methods used there be employed in § 954? Is the method here suggested preferable? why?

956. Def. Two **spherical polygons** are **symmetrical** if the corresponding polyhedral angles are symmetrical.



957. The following are some of the properties of symmetrical spherical triangles; the student should prove the correctness of each:

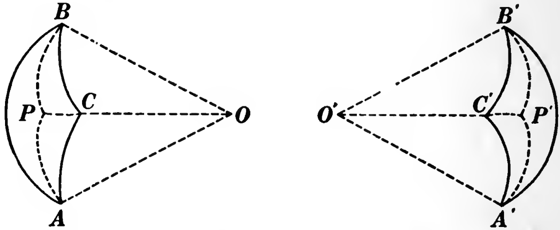
(*a*) *Symmetrical spherical triangles have their parts respectively equal, but arranged in reverse order.*

(*b*) *Two isosceles symmetrical spherical triangles are equal.*

Hint. Prove (*b*) by superposition or by showing that the corresponding trihedral ∠s are equal.

Proposition XV. Theorem

958. *In equal spheres, or in the same sphere, two symmetrical spherical triangles are equivalent.*



Given symmetrical spherical $\triangle$ ABC and $A'B'C'$ in equal spheres O and O'.

To prove spherical $\triangle ABC \approx$ spherical $\triangle A'B'C'$.

Argument	Reasons
1. Let P and P' be poles of small ⓢ through A, B, C, and A', B', C', respectively.	1. § 908, *h*.
2. Arcs AB, BC, CA are equal, respectively, to arcs $A'B'$, $B'C'$, $C'A'$.	2. § 957, *a*.
3. $\therefore$ chords AB, BC, CA, are equal, respectively, to chords $A'B'$, $B'C'$, $C'A'$.	3. § 298, II.
4. $\therefore$ plane $\triangle ABC =$ plane $\triangle A'B'C'$.	4. § 116.
5. $\therefore \odot ABC = \odot A'B'C'$.	5. § 324.
6. Draw arcs of great ⓢ PA, PB, PC, $P'A'$, $P'B'$, and $P'C'$.	6. § 908, *g*.
7. Then $\overset{\frown}{PA} = \overset{\frown}{PB} = \overset{\frown}{PC} = \overset{\frown}{P'A'} = \overset{\frown}{P'B'} = \overset{\frown}{P'C'}$.	7. § 913.
8. $\therefore$ isosceles spherical $\triangle$ APB and $A'P'B'$ are symmetrical.	8. § 956.
9. $\therefore \triangle APB = \triangle A'P'B'$.	9. § 957, *b*.
10. Likewise $\triangle BPC = \triangle B'P'C'$ and $\triangle CPA = \triangle C'P'A'$.	10. By steps similar to 8–9.
11. $\therefore$ spherical $\triangle ABC \approx$ spherical $\triangle A'B'C'$.	11. § 54, 2.

Q.E.D.

PROPOSITION XVI. THEOREM

959. *In equal spheres, or in the same sphere, two spherical triangles are symmetrical, and therefore equivalent:*

I. *If a side and the two adjacent angles of one are equal respectively to a side and the two adjacent angles of the other;*

II. *If two sides and the included angle of one are equal respectively to two sides and the included angle of the other;*

III. *If the three sides of one are equal respectively to the three sides of the other:*

provided the equal parts are arranged in reverse order.

The proofs are left as exercises for the student.

HINT. In each of the above cases prove the corresponding trihedral ∠s symmetrical (§ 709); and thus show that the spherical △s are symmetrical.

Ex. 1526. The bisector of the angle at the vertex of an isosceles spherical triangle is perpendicular to the base and bisects it.

Ex. 1527. The arc drawn from the vertex of an isosceles spherical triangle to the mid-point of the base bisects the vertex angle and is perpendicular to the base.

Ex. 1528. State and prove the theorems on the sphere corresponding to the following theorems on the plane:

(1) Every point in the perpendicular bisector of a line is equidistant from the ends of that line (§ 134).

(2) Every point equidistant from the ends of a line lies in the perpendicular bisector of that line (§ 139).

(3) Every point in the bisector of an angle is equidistant from the sides of the angle (§ 253).

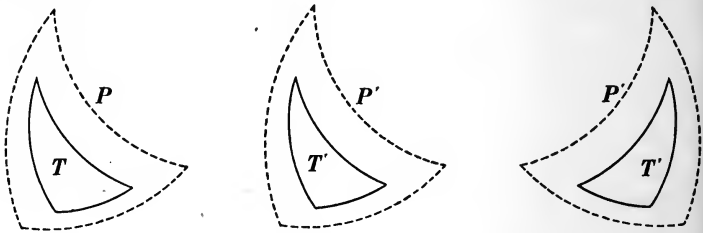
HINT. In the figure for (3), corresponding to the figure of § 252, draw $\widehat{PD} \perp \widehat{AB}$ and lay off $\widehat{BE} = \widehat{BD}$.

Ex. 1529. The diagonals of an equilateral spherical quadrilateral are perpendicular to each other. Prove. State the theorem in plane geometry that corresponds to this exercise.

Ex. 1530. In equal spheres, or in the same sphere, if two spherical triangles are mutually equilateral, their polar triangles are mutually equiangular; and conversely.

PROPOSITION XVII. THEOREM

960. *In equal spheres, or in the same sphere, if two spherical triangles are mutually equiangular, they are mutually equilateral, and are either equal or symmetrical.*



Given spherical $\triangle$ T and T' in equal spheres, or in the same sphere, and mutually equiangular.

To prove T and T' mutually equilateral and either equal or symmetrical.

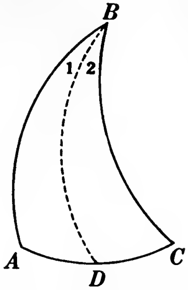
ARGUMENT	REASONS
1. Let $\triangle$ P and P' be the polars of $\triangle$ T and T', respectively.	1. § 943.
2. T and T' are mutually equiangular.	2. By hyp.
3. Then P and P' are mutually equilateral.	3. § 947.
4. $\therefore$ P and P' are either equal or symmetrical, and hence are mutually equiangular.	4. §§ 954, III, and 959, III.
5. $\therefore$ T and T' are mutually equilateral.	5. § 947.
6. $\therefore$ T and T' are either equal or symmetrical. Q.E.D.	6. Same reason as 4.

Ex. 1531. In plane geometry, if two triangles are mutually equiangular, what can be said of them? Are they equal? equivalent?

Ex. 1532. Find the locus of all points of a sphere that are equidistant from two given points on the surface of the sphere; from two given points in space, not on the surface of the sphere.

PROPOSITION XVIII. THEOREM

961. *The base angles of an isosceles spherical triangle are equal.*



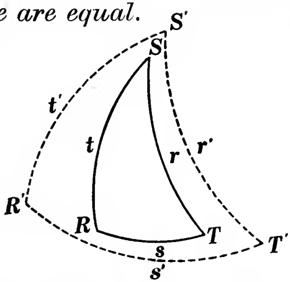
Given isosceles spherical $\triangle ABC$, with side $AB =$ side BC.

To prove $\angle A = \angle C$.

HINT. Compare with § 111.

PROPOSITION XIX. THEOREM

962. *If two angles of a spherical triangle are equal, the sides opposite are equal.*



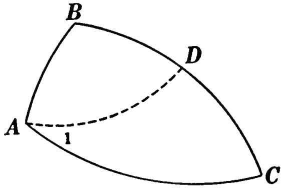
Given spherical $\triangle RST$ with $\angle R = \angle T$.

To prove $r = t$.

ARGUMENT	REASONS
1. Let $\triangle R'S'T'$ be the polar of $\triangle RST$.	1. § 943.
2. $\angle R = \angle T$.	2. By hyp.
3. $\therefore r' = t'$.	3. § 947.
4. $\therefore R' = T'$.	4. § 961.
5. $\therefore r = t$. Q.E.D.	5. § 947.

Proposition XX. Theorem

963. *If two angles of a spherical triangle are unequal, the side opposite the greater angle is greater than the side opposite the less angle.*



Given spherical $\triangle ABC$ with $\angle A > \angle C$.

To prove $\overset{\frown}{BC} > \overset{\frown}{AB}$.

Argument	Reasons
1. Draw an arc of a great $\odot$ AD making $\angle 1 = \angle C$.	1. § 908, *g*.
2. Then $\overset{\frown}{AD} = \overset{\frown}{DC}$.	2. § 962.
3. But $\overset{\frown}{BD} + \overset{\frown}{AD} > \overset{\frown}{AB}$.	3. § 941.
4. $\therefore$ $\overset{\frown}{BD} + \overset{\frown}{DC} > \overset{\frown}{AB}$; *i.e.* $\overset{\frown}{BC} > \overset{\frown}{AB}$. Q.E.D.	4. § 309.

Ex. 1533. In a birectangular spherical triangle the side included by the two right angles is less than, equal to, or greater than, either of the other two sides, according as the angle opposite is less than, equal to, or greater than 90°.

Ex. 1534. An equilateral spherical triangle is also equiangular.

Ex. 1535. If two face angles of a trihedral angle are equal, the dihedral angles opposite are equal.

Ex. 1536. State and prove the converse of Ex. 1534.

Ex. 1537. State and prove the converse of Ex. 1535.

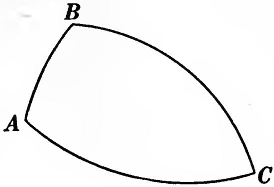
Ex. 1538. The arcs bisecting the base angles of an isosceles spherical triangle form an isosceles spherical triangle.

Ex. 1539. The bases of an isosceles trapezoid are 14 inches and 6 inches and the altitude 3 inches; find the total area and volume of the solid generated by revolving the trapezoid about its longer base as an axis.

Ex. 1540. Find the total area and volume of the solid generated by revolving the trapezoid of Ex. 1539 about its shorter base as an axis.

PROPOSITION XXI. THEOREM

964. *If two sides of a spherical triangle are unequal, the angle opposite the greater side is greater than the angle opposite the less side.*



Given spherical $\triangle ABC$ with $\widehat{BC} > \widehat{AB}$.

To prove $\angle A > \angle C$.

The proof is left as an exercise for the student.

HINT. Prove by the indirect method.

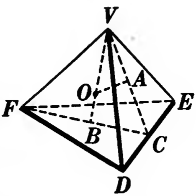
965. Questions. Could Prop. XXI have been proved by the method used in § 156? Does reason 4 of that proof hold in spherical ⊿? See Ex. 1525.

Ex. 1541. If two adjacent sides of a spherical quadrilateral are greater, respectively, than the other two sides, the spherical angle included between the two shorter sides is greater than the spherical angle between the two greater sides.

HINT. Compare with Ex. 153.

Ex. 1542. Find the total area and volume of the solid generated by revolving the trapezoid of Ex. 1539 about the perpendicular bisector of its bases as an axis.

Ex. 1543. Find the radius of the sphere inscribed in a regular tetrahedron whose edge is a.



HINT. Let O be the center of the sphere, A the center of face VED, and B the center of face EDF. Then OA = radius of inscribed sphere. Show that rt. ⊿ VAO and VBC are similar. Then $VO : VC = VA : VB$. VC, VA, and VB can be found (Ex. 1328). Find VO, then OA.

Ex. 1544. Find the radius of the sphere circumscribed about a regular tetrahedron whose edge is a.

HINT. In the figure of Ex. 1543, draw AD and OD.

MENSURATION OF THE SPHERE

AREAS

PROPOSITION XXII. THEOREM

966. *If an isosceles triangle is revolved about a straight line lying in its plane and passing through its vertex but not intersecting its surface, the area of the surface generated by the base of the triangle is equal to the product of its projection on the axis and the circumference of a circle whose radius is the altitude of the triangle.*

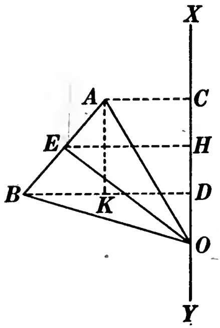


FIG. 1.

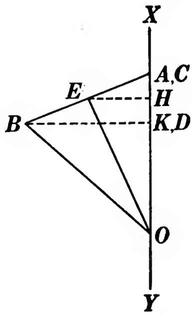


FIG. 2.

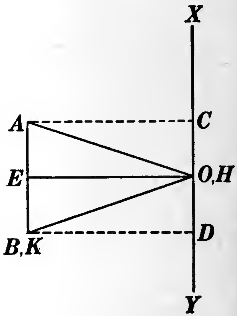


FIG. 3.

Given isosceles $\triangle AOB$ with base AB and altitude OE, a str. line XY lying in the plane of $\triangle AOB$ passing through O and not intersecting the surface of $\triangle AOB$, and CD the projection of AB on XY; let the area of the surface generated by AB be denoted by area AB.

To prove area $AB = CD \cdot 2\,\pi OE$.

I. If AB is not $\parallel XY$ and does not meet XY (Fig. 1).

ARGUMENT ONLY

1. From E draw $EH \perp XY$.

2. Since the surface generated by AB is the surface of a frustum of a rt. circular cone, area $AB = AB \cdot 2\,\pi EH$.

3. From A draw $AK \perp BD$.

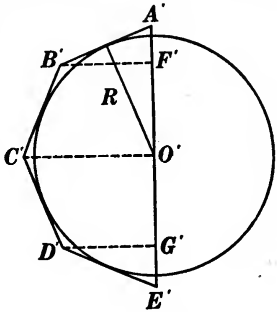
4. Then in rt. $\triangle$ BAK and OEH, $\angle BAK = \angle OEH$.

5. $\therefore \triangle BAK \sim \triangle OEH.$
6. $\therefore AB : AK = OE : EH$; *i.e.* $AB \cdot EH = AK \cdot OE.$
7. But $AK = CD$; $\therefore AB \cdot EH = CD \cdot OE.$
8. $\therefore$ area $AB = CD \cdot 2\,\pi OE.$ Q.E.D.

II. If AB is not $\parallel XY$ and point A lies in XY (Fig. 2).
III. If $AB \parallel XY$ (Fig. 3).
The proofs of II and III are left as exercises for the student.

HINT. See if the proof given for I will apply to Figs. 2 and 3.

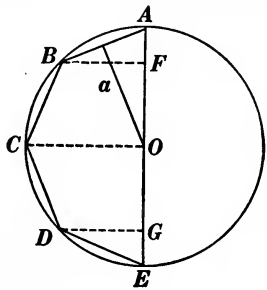
967. Cor. I. *If half of a regular polygon with an even number of sides is circumscribed about a semicircle, the area of the surface generated by its semiperimeter as it revolves about the diameter of the semicircle as an axis, is equal to the product of the diameter of the regular polygon and the circumference of a circle whose radius is* R, *the radius of the given semicircle.*



OUTLINE OF PROOF

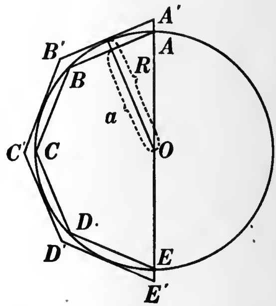
1. Area $A'B' = A'F' \cdot 2\,\pi R$;
   area $B'C' = F'O' \cdot 2\,\pi R$; etc.
2. $\therefore$ area $A'B'C' \cdots = (A'F' + F'O' + \cdots)\,2\,\pi R = A'E' \cdot 2\,\pi R.$

968. Cor. II. *If half of a regular polygon with an even number of sides is inscribed in a semicircle, the area of the surface generated by its semiperimeter as it revolves about the diameter of the semicircle as an axis, is equal to the product of the diameter of the semicircle and the circumference of a circle whose radius is the apothem of the regular polygon.*



HINT. Prove area $ABC \cdots = AE \cdot 2\,\pi a.$

969. Cor. III. *If halves of regular polygons with the same even number of sides are circumscribed about, and inscribed in, a semicircle, then by repeatedly doubling the number of sides of these polygons, and making the polygons always regular, the surfaces generated by the semiperimeters of the polygons as they revolve about the diameter of the semicircle as an axis approach a common limit.*



Outline of Proof

1. If S and s denote the areas of the surfaces generated by the semiperimeters $A'B'C'\cdots$ and $ABC\cdots$ as they revolve about $A'E'$ as an axis,

then $S = A'E' \cdot 2\,\pi R$ (§ 967);

and $s = AE \cdot 2\,\pi a$ (§ 968).

2. $\therefore \dfrac{S}{s} = \dfrac{A'E' \cdot 2\,\pi R}{AE \cdot 2\,\pi a} = \dfrac{A'E'}{AE} \cdot \dfrac{R}{a}.$

3. But polygon $A'B'C'\cdots \sim$ polygon $ABC\cdots$ (§ 438).

4. $\therefore \dfrac{A'E'}{AE} = \dfrac{A'B'}{AB} = \dfrac{R}{a}$ (§§ 419, 435).

5. $\therefore \dfrac{S}{s} = \dfrac{R}{a} \cdot \dfrac{R}{a} = \dfrac{R^2}{a^2}.$

6. $\therefore \dfrac{S-s}{S} = \dfrac{R^2-a^2}{R^2}.$

7. $\therefore$ by steps similar to Args. 6–11 (§ 855), S and s approach a common limit. Q.E.D.

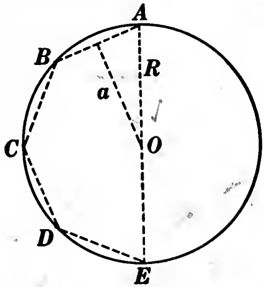
970. Def. The **surface of a sphere** is the common limit which the successive surfaces generated by halves of regular polygons with the same even number of sides approach, if these semipolygons fulfill the following conditions:

(1) They must be circumscribed about, and inscribed in, a great semicircle of the sphere;

(2) The number of sides must be successively increased, each side approaching zero as a limit.

Proposition XXIII. Theorem

971. *The area of the surface of a sphere is equal to four times the area of a great circle of the sphere.*



Given sphere O with its radius denoted by R, and the area of its surface denoted by S.

To prove $S = 4\,\pi R^2$.

Argument	Reasons
1. In the semicircle ACE inscribe $ABCDE$, half of a regular polygon with an even number of sides. Denote its apothem by a, and the area of the surface generated by the semiperimeter as it revolves about AE as an axis by S'.	1. § 517, *a*.
2. Then $S' = AE \cdot 2\,\pi a$; *i.e.* $S' = 2\,R \cdot 2\,\pi a = 4\,\pi Ra$.	2. § 968.
3. As the number of sides of the regular polygon, of which $ABCDE$ is half, is repeatedly doubled, S' approaches S as a limit.	3. § 970.
4. Also a approaches R as a limit.	4. § 543, I.
5. $\therefore 4\,\pi R \cdot a$ approaches $4\,\pi R \cdot R$, *i.e.* $4\,\pi R^2$, as a limit.	5. § 590.
6. But S' is always equal to $4\,\pi R \cdot a$.	6. Arg. 2.
7. $\therefore S = 4\,\pi R^2$. Q.E.D.	7. § 355.

972. Cor. I. *The areas of the surfaces of two spheres are to each other as the squares of their radii and as the squares of their diameters.*

OUTLINE OF PROOF

1. $S = 4\,\pi R^2$ and $S' = 4\,\pi R'^2$; $\therefore \dfrac{S}{S'} = \dfrac{4\,\pi R^2}{4\,\pi R'^2} = \dfrac{R^2}{R'^2}$.

2. But $\dfrac{R^2}{R'^2} = \dfrac{4\,R^2}{4\,R'^2} = \dfrac{(2\,R)^2}{(2\,R')^2} = \dfrac{D^2}{D'^2}$; $\therefore \dfrac{S}{S'} = \dfrac{D^2}{D'^2}$.

973. Historical Note. Prop. XXIII is given as Prop. XXXV in the treatise entitled *Sphere and Cylinder* by Archimedes, already spoken of in § 809.

Ex. 1545. Find the surface of a sphere whose diameter is 16 inches.

Ex. 1546. What will it cost to gild the surface of a globe whose radius is $1\frac{1}{2}$ decimeters, at an average cost of $\frac{2}{5}$ of a cent per square centimeter?

Ex. 1547. The area of a section of a sphere made by a plane 11 inches from the center is $3600\,\pi$ square inches. Find the surface of the sphere.

Ex. 1548. Find the surface of a sphere circumscribed about a cube whose edge is 12 inches.

Ex. 1549. The radius of a sphere is R. Find the radius of a sphere whose surface is twice the surface of the given sphere; one half; one nth.

Ex. 1550. Find the surface of a sphere whose diameter is $2\,R$, and the total surface of a right circular cylinder whose altitude and diameter are each equal to $2\,R$.

Ex. 1551. From the results of Ex. 1550 state, in the form of a theorem, the relation of the surface of a sphere to the total surface of the circumscribed cylinder.

Ex. 1552. Show that, in Ex. 1550, the surface of the sphere is exactly equal to the lateral surface of the cylinder.

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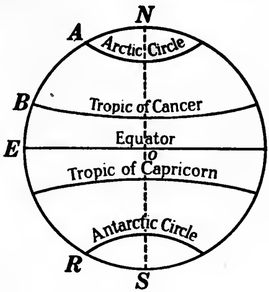
974. Historical Note. The discovery of the remarkable property that the surface of a sphere is two thirds of the surface of the circumscribed cylinder (Exs. 1550 and 1551) is due again to Archimedes. The discovery of this proposition, and the discovery of the corresponding proposition for volumes (§ 1001), were the philosopher's chief pride, and he therefore asked that a figure of this proposition be inscribed on his tomb. His wishes were carried out by his friend Marcellus. (For a further account of Archimedes, read also § 542.)

975. Defs. A **zone** is a closed figure on the surface of a sphere whose boundary is composed of the circumferences of two circles whose planes are parallel.

The circumferences forming the boundary of a zone are its **bases**.

Thus, if semicircle NES is revolved about NS as an axis, arc AB will generate a zone, while points A and B will generate the bases of the zone.



976. Def. The **altitude** of a zone is the perpendicular from any point in the plane of one base to the plane of the other base.

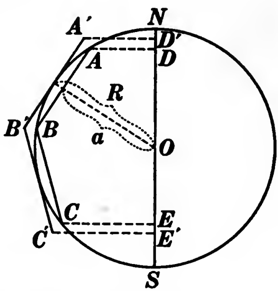
977. Def. If the plane of one of the bases of a zone is tangent to the sphere, the zone is called a **zone of one base.**

Thus, arc NA or arc RS will generate a zone of one base.

978. Questions. Is the term "zone" used in exactly the same sense here as it is in the geography? Name the geographical zones of one base; of two bases. Name the five circles whose circumferences form the bases of the six geographical zones. Which of these are great circles?*

979. Cor. II. *The area of a zone is equal to the product of its altitude and the circumference of a great circle.*

Outline of Proof



Let S denote the area generated by broken line $A'B'C'$, s by broken line ABC, and Z by arc ABC; let DE, the altitude of the zone, be denoted by H.

Then $S = D'E' \cdot 2\pi R$;

$s = DE \cdot 2\pi a$.

$\therefore \dfrac{S}{s} = \dfrac{R^2}{a^2}$. (See Args. 2–5, § 969.) Then by steps similar to §§ 969–971, $Z = H \cdot 2\pi R$.

* The student will observe that the projection used in this figure is different from that used in the other figures.

980. Cor. III. *In equal spheres, or in the same sphere, the areas of two zones are to each other as their altitudes.*

981. Question. In general, surfaces are to each other as the products of two lines. Is § 980 an exception to this rule? Explain.

Ex. 1553. The area of a zone of one base is equal to the area of a circle whose radius is the chord of the arc generating the zone.

HINT. Use §§ 979 and 444, II.

Ex. 1554. Show that the formula of § 971 is a special case of § 979.

Ex. 1555. Find the area of the surface of a zone if the distance between its bases is 8 inches and the radius of the sphere is 6 inches.

Ex. 1556. The diameter of a sphere is 16 inches. Three parallel planes divide this diameter into four equal parts. Find the area of each of the four zones thus formed.

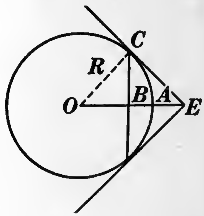
Ex. 1557. Prove that one half of the earth's surface lies within 30° of the equator.

Ex. 1558. Considering the earth as a sphere with radius R, find the area of the zone adjoining the north pole, whose altitude is $\frac{R}{3}$; $\frac{2R}{3}$. Is the one area twice the other?

Ex. 1559. Considering the earth as a sphere with radius R, find the area of the zone extending 30° from the north pole; 60° from the north pole. Is the one area twice the other?

Ex. 1560. Considering the earth as a sphere with radius R, find the area of the zone whose bases are parallels of latitude: (*a*) 30° and 45° from the north pole; (*b*) 30° and 45° from the equator. Are the two areas equal? Explain your answer.

Ex. 1561. How far from the center of a sphere whose radius is R must the eye of an observer be so that one sixth of the surface of the sphere is visible?

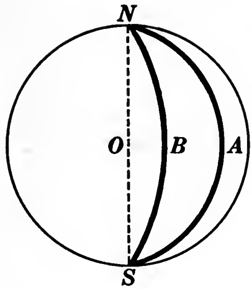


HINT. Let E be the eye of the observer. Then AB must $=\frac{R}{3}$. Find OB, then use § 443, II.

Ex. 1562. What portion of the surface of a sphere can be seen if the distance of the eye of the observer from the center of the sphere is $2R$? $3R$? nR?

Ex. 1563. The radii of two concentric spheres are 6 inches and 10 inches. A plane is passed tangent to the inner sphere. Find: (*a*) the area of the section of the outer sphere made by the plane; (*b*) the area of the surface cut off of the outer sphere by the plane.

982. Def. A **lune** is a closed figure on the surface of a sphere whose boundary is composed of two semicircumferences of great circles, as *NASB*.



983. Defs. The two semicircumferences are called the **sides** of the lune, as *NAS* and *NBS*; the points of intersection of the sides are called the **vertices** of the lune, as *N* and *S*; the spherical angles formed at the vertices by the sides of the lune are called the **angles of the lune**, as $\angle$s *ANB* and *BSA*.

984. Prove, by superposition, the following property of lunes: *In equal spheres, or in the same sphere, two lunes are equal if their angles are equal.*

985. So far, the surfaces considered in connection with the sphere have been measured in terms of square units, *i.e.* square inches, square feet, etc. For example, if the radius of a sphere is 6 inches, the surface of the sphere is $4\pi 6^2$, *i.e.* 144π square inches. But, as the sides and angles of a lune and a spherical polygon are given in degrees and not in linear units, it will be necessary to introduce some new unit for determining the areas of these figures. For this purpose the entire surface of a sphere is thought of as being divided into 720 equal parts, and each one of these parts is called a spherical degree. Hence:

986. Def. A **spherical degree** is $\frac{1}{720}$ of the surface of a sphere. Now if the area of a lune or of a spherical triangle can be obtained in spherical degrees, the area can easily be changed to square units. For example, if it is found that the area of a spherical triangle is 80 spherical degrees, its area is $\frac{80}{720}$, *i.e.* $\frac{1}{9}$ of the entire surface of the sphere. On the sphere whose radius is 6 inches, the area of the given triangle will be $\frac{1}{9}$ of 144π square inches, *i.e.* 16π square inches. The following theorems are for the purpose of determining the areas of figures on the surface of a sphere in terms of spherical degrees.

PROPOSITION XXIV. THEOREM

987. *The area of a lune is to the area of the surface of the sphere as the number of degrees in the angle of the lune is to 360.*

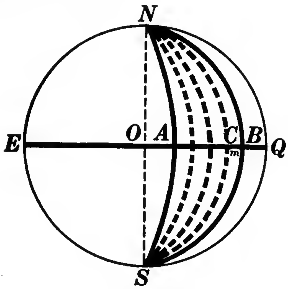


FIG. 1.

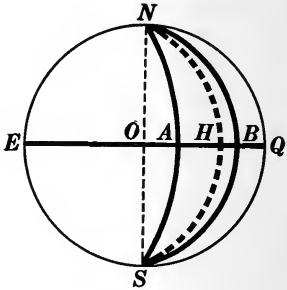


FIG. 2.

Given lune $NASB$ with the number of degrees in its $\angle$ denoted by N, its area denoted by L, and the area of the surface of the sphere denoted by S; let $\odot EQ$ be the great $\odot$ whose pole is N.

To prove $\dfrac{L}{S} = \dfrac{N}{360}$.

I. If arc AB and circumference EQ are commensurable (Fig. 1).

ARGUMENT	REASONS
1. Let m be a common measure of arc AB and circumference EQ, and suppose that m is contained in arc AB r times and in circumference EQ t times.	1. § 335.
2. Then $\dfrac{\text{arc } AB}{\text{circumference } EQ} = \dfrac{r}{t}$.	2. § 341.
3. Through the several points of division on circumference EQ pass semicircumferences of great circles from N to S.	3. § 908, *h*.
4. Then lune $NASB$ is divided into r lunes and the surface of the sphere into t lunes, each equal to lune $NCSB$.	4. § 984.

Argument	Reasons
5. $\therefore \frac{L}{S} = \frac{r}{t}$.	5. § 341.
6. $\therefore \frac{L}{S} = \frac{\text{arc } AB}{\text{circumference } EQ}$.	6. § 54, 1.
7. But arc AB is the measure of $\angle N$; *i.e.* it contains N degrees.	7. § 918.
8. And circumference EQ contains 360°.	8. § 297.
9. $\therefore \frac{L}{S} = \frac{N}{360}$. Q.E.D.	9. § 309.

II. If arc AB and circumference EQ are incommensurable (Fig. 2).

The proof is left as an exercise for the student.

HINT. The proof is similar to that of § 409, II.

988. Cor. I. *The area of a lune, expressed in spherical degrees, is equal to twice the number of degrees in its angle.*

OUTLINE OF PROOF

$$\frac{L}{S} = \frac{N}{360} \text{ (§ 987).} \quad \therefore \frac{L}{720} = \frac{N}{360}. \quad \therefore L = 2N.$$

Ex. 1564. Find the area of a lune in spherical degrees if its angle is 35°. What part is the lune of the entire surface of the sphere?

Ex. 1565. Find the area of a lune in square inches if its angle is 42° and the radius of the sphere is 8 inches. (Use $\pi = \frac{22}{7}$.)

Ex. 1566. In equal spheres, or in the same sphere, two lunes are to each other as their angles.

Ex. 1567. Two lunes in unequal spheres, but with equal angles, are to each other as the squares of the radii of their spheres.

Ex. 1568. In a sphere whose radius is R, find the altitude of a zone equivalent to a lune whose angle is 45°.

Ex. 1569. Considering the earth as a sphere with radius R, find the area of the zone visible from a point at a height h above the surface of the earth.

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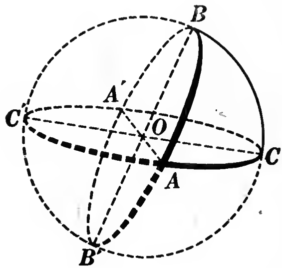
989. Def. The **spherical excess** of a spherical triangle is the excess of the sum of its angles over 180°.

Proposition XXV. Theorem

990. *The area of a spherical triangle, expressed in spherical degrees, is equal to its spherical excess.*



Given spherical $\triangle ABC$ with its spherical excess denoted by E.

To prove area of $\triangle ABC = E$ spherical degrees.

Argument	Reasons
1. Complete the circumferences of which AB, BC, and CA are arcs.	1. § 908, *g*.
2. $\triangle$s $AB'C'$ and $A'BC$ are symmetrical.	2. § 956.
3. $\therefore \triangle AB'C' \Leftrightarrow \triangle A'BC$.	3. § 958.
4. $\therefore \triangle ABC + \triangle AB'C' \Leftrightarrow \triangle ABC + \triangle A'BC$.	4. § 54, 2.
5. $\therefore$, expressed in spherical degrees, $\triangle ABC + \triangle AB'C' \Leftrightarrow$ lune $A = 2A$; $\triangle ABC + \triangle AB'C =$ lune $B = 2B$; $\triangle ABC + \triangle ABC' =$ lune $C = 2C$.	5. § 988.
6. $\therefore 2\triangle ABC + (\triangle ABC + \triangle AB'C' + \triangle AB'C + \triangle ABC') = 2(A+B+C)$.	6. § 54, 2.
7. But $\triangle ABC + \triangle AB'C' + \triangle AB'C + \triangle ABC'$ = surface of a hemisphere = 360.	7. § 985.
8. $\therefore 2\triangle ABC + 360 = 2(A+B+C)$.	8. § 309.
9. $\therefore \triangle ABC + 180 = A + B + C$.	9. § 54, 8 *a*.
10. $\therefore \triangle ABC = (A+B+C) - 180$, *i.e.* E spherical degrees. Q.E.D.	10. § 54, 3.

991. In § 949 it was proved that the sum of the angles of a spherical triangle is greater than 180° and less than 540°. Hence the spherical excess of a spherical triangle may vary from 0° to 360°, from which it follows (§ 990) that the area of a spherical triangle may vary from $\frac{0}{720}$ to $\frac{360}{720}$ of the entire surface; *i.e.* the area of a spherical triangle may vary from nothing to $\frac{1}{2}$ the surface of the sphere. Thus in a spherical triangle whose angles are 70°, 80°, and 100°, respectively, the spherical excess is $(70° + 80° + 100°) - 180° = 70°$; *i.e.* the area of the given triangle is $\frac{70}{720}$ of the surface of the sphere.

992. Historical Note. Menelaus of Alexandria (*circ.* 98 A.D.) wrote a treatise in which he describes the properties of spherical triangles, although there is no attempt at their solution. The expression for the area of a spherical triangle, as stated in § 990, was first given about 1626 A.D. by Girard. (See also § 946.) This theorem was also discovered independently by Cavalieri, a prominent Italian mathematician.

Ex. 1570. If three great circles are drawn, each perpendicular to the other two, into how many trirectangular spherical triangles is the surface divided? Then what is the area of a trirectangular spherical triangle in spherical degrees? Test your answer by applying Prop. XXV.

Ex. 1571. Find the area in spherical degrees of a birectangular spherical triangle one of whose angles is 70°; of an equilateral spherical triangle one of whose angles is 80°. What part of the surface of the sphere is each triangle?

Ex. 1572. The angles of a spherical triangle in a sphere whose surface has an area of 216 square feet are 95°, 105°, and 130°. Find the number of square feet in the area of the triangle.

Ex. 1573. In a sphere whose diameter is 16 inches, find the area of a triangle whose angles are 70°, 86°, and 120°.

Ex. 1574. The angles of a spherical triangle are 60°, 120°, and 160°, and its area is $100\frac{4}{7}$ square inches. Find the radius of the sphere. (Use $\pi = \frac{22}{7}$.)

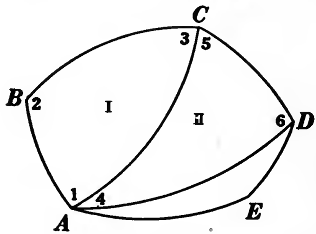
Ex. 1575. The area of a spherical triangle is 90 spherical degrees, and the angles are in the ratio of 2, 3, and 5. Find the angles.

Ex. 1576. Find the angle (1) of an equilateral spherical triangle, (2) of a lune, each equivalent to one third the surface of a sphere.

Ex. 1577. Find the angle of a lune equivalent to an equilateral spherical triangle one of whose angles is 84°.

Proposition XXVI. Theorem

993. *The area of a spherical polygon, expressed in spherical degrees, is equal to the sum of its angles diminished by 180° taken as many times less two as the polygon has sides.*



Given spherical polygon $ABCD \cdots$ with n sides; denote the sum of its angles by T.

To prove area of polygon $ABCD \cdots$, expressed in spherical degrees, $= T-(n-2)180$.

Argument	Reasons
1. From any vertex such as A, draw all possible diagonals of the polygon, forming $n-2$ spherical ▵, I, II, etc.	1. § 937.
2. Then, expressed in spherical degrees, $\triangle \mathrm{I} = (\angle 1 + \angle 2 + \angle 3) - 180$; $\triangle \mathrm{II} = (\angle 4 + \angle 5 + \angle 6) - 180$; etc.	2. § 990.
3. $\therefore \triangle \mathrm{I} + \triangle \mathrm{II} + \cdots = T-(n-2)180$.	3. § 54, 2.
4. $\therefore$ area of polygon $ABCD \cdots$ $= T-(n-2)\,180$. Q.E.D.	4. § 309.

Ex. 1578. Prove Prop. XXVI by using a figure similar to that used in § 216.

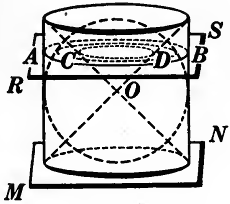
Ex. 1579. Find the area of a spherical polygon whose angles are 80°, 92°, 120°, and 140°, in a sphere whose radius is 8 inches.

Ex. 1580. Find the angle of an equilateral spherical triangle equivalent to a spherical pentagon whose angles are 90°, 100°, 110°, 130°, and 140°.

Ex. 1581. Find one angle of an equiangular spherical hexagon equivalent to six equilateral spherical triangles each with angles of 70°.

Ex. 1582. The area of a section of a sphere 63 inches from the center is $256\,\pi$ square inches. Find the surface of the sphere.

Ex. 1583. The figure represents a sphere inscribed in a cylinder, and two cones with the bases of the cylinder as their bases and the center of the sphere as their vertices. Any plane, as RS, is passed through the figure parallel to MN, the plane of the base. Prove that the ring between section AB of the cylinder, and section CD of the cone, is always equivalent to the section of the sphere.



Ex. 1584. Find the volume of a barrel 30 inches high, 54 inches in circumference at the top and bottom, and 64 inches in circumference at the middle.

HINT. Consider the barrel as the sum of two frustums of cones.

Ex. 1585. Given T the total area, and R the radius of the base, of a right circular cylinder. Find the altitude.

Ex. 1586. Given S the lateral area, and R the radius of the base, of a right circular cone. Find the volume.

Ex. 1587. Given S the lateral area, and T the total area, of a right circular cone. Find the radius and the altitude.

VOLUMES

994. Note. The student should not fail to observe the striking similarity in the figures and theorems, as well as in the definitions, relating to the areas and volumes connected with the measurement of the sphere. A careful comparison of the following articles will emphasize this similarity:

AREAS	VOLUMES	AREAS	VOLUMES
§ 966	§ 995	§ 979	§§ 1004, 1005
§§ 967, 968	§ 996, *a*	§ 982	§ 1006
§ 969	§ 996, *b*	§ 984	§ 1007
§ 970	§ 996, *c*	§ 987	§ 1008
§ 971	§ 997	§ 988	§ 1009
§ 972	§ 999	§ 936	§ 1010
§ 975	§ 1002	§ 990	§ 1012
§ 977	§ 1003	§ 993	§ 1013

Proposition XXVII. Theorem

995. *If an isosceles triangle is revolved about a straight line lying in its plane and passing through its vertex but not intersecting its surface, the volume of the solid generated is equal to the product of the surface generated by the base of the triangle and one third of its altitude.*

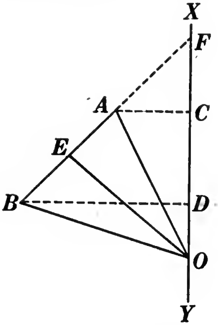


Fig. 1.

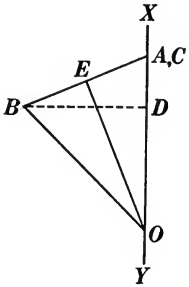


Fig. 2.

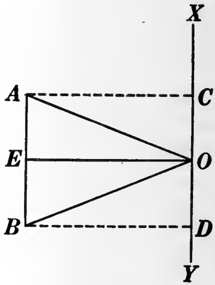


Fig. 3.

Given isosceles $\triangle AOB$ with altitude OE, and a str. line XY lying in the plane of $\triangle AOB$, passing through O and not intersecting the surface of $\triangle AOB$; let the volume of the solid generated by $\triangle AOB$ revolving about XY as an axis be denoted by volume AOB.

To prove volume $AOB =$ area $AB \cdot \frac{1}{3} OE$.

I. If AB is not $\parallel XY$ and does not meet XY (Fig. 1).

Argument Only

1. Draw AC and $BD \perp XY$.
2. Prolong BA to meet XY at F.
3. Then volume $AOB =$ volume $FOB -$ volume FOA.
4. Volume $FOB =$ volume $FDB +$ volume DOB.
5. $\therefore$ volume $FOB = \frac{1}{3}\pi\overline{BD}^2 \cdot FD + \frac{1}{3}\pi\overline{BD}^2 \cdot DO$
$= \frac{1}{3}\pi\overline{BD}^2(FD + DO) = \frac{1}{3}\pi BD \cdot BD \cdot FO.$
6. But $BD \cdot FO =$ twice area of $\triangle FOB = BF \cdot OE$.
7. $\therefore$ volume $FOB = \frac{1}{3}\pi BD \cdot BF \cdot OE = \pi BD \cdot BF \cdot \frac{1}{3} OE$.
8. But $\pi BD \cdot BF =$ area FB.
9. $\therefore$ volume $FOB =$ area $FB \cdot \frac{1}{3} OE$.

10. Likewise volume $FOA =$ area $FA \cdot \frac{1}{3} OE$.

11. $\therefore$ volume $AOB =$ area $FB \cdot \frac{1}{3} OE -$ area $FA \cdot \frac{1}{3} OE$

$= (\text{area } FB - \text{area } FA) \frac{1}{3} OE$

$= \text{area } AB \cdot \frac{1}{3} OE$. Q.E.D.

II. If AB is not $\parallel XY$ and point A lies in XY (Fig. 2).

The proof is left as an exercise for the student.

HINT. See Arg. 9 or Arg. 10 of § 995, I.

III. If $AB \parallel XY$ (Fig. 3).

The proof is left as an exercise for the student.

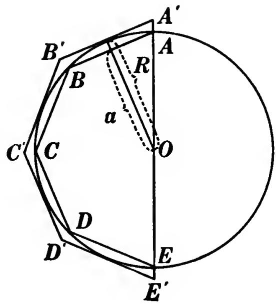
HINT. Volume $AOB =$ volume $ACDB -$ twice volume COA.

996. The student may:

(*a*) State and prove the corollaries on the volume of a sphere corresponding to §§ 967 and 968.

(*b*) State and prove the theorem on the volume of a sphere corresponding to § 969.

OUTLINE OF PROOF



1. $V = (\text{area } A'B'C' \cdots) \frac{1}{3} R$

$= A'E' \cdot 2\pi R \cdot \frac{1}{3} R$

$= \frac{2}{3}\pi R^2 \cdot A'E'$

(§§ 996, *a* and 967).

2. $v = (\text{area } ABC \cdots) \frac{1}{3} a$

$= AE \cdot 2\pi a \cdot \frac{1}{3} a$

$= \frac{2}{3}\pi a^2 AE$

(§§ 996, *a* and 968).

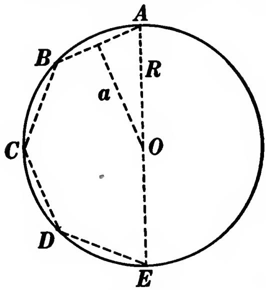
3. $\therefore \dfrac{V}{v} = \dfrac{\frac{2}{3}\pi R^2 \cdot A'E'}{\frac{2}{3}\pi a^2 \cdot AE} = \dfrac{R^2}{a^2} \cdot \dfrac{A'E'}{AE} = \dfrac{R^3}{a^3}$.

4. Proceed as in § 855, observing that since the limit of $a = R$ (§ 543, I), the limit of $a^3 = R^3$ (§ 593); *i.e.* $R^3 - a^3$ may be made less than any previously assigned value, however small.

(*c*) State, by aid of § 970, the definition of the **volume of a sphere.**

PROPOSITION XXVIII. THEOREM

997. *The volume of a sphere is equal to the product of the area of its surface and one third its radius.*



Given sphere O with its radius denoted by R, the area of its surface by S, and its volume by V.

To prove $V = S \cdot \frac{1}{3} R$.

ARGUMENT	REASONS
1. In the semicircle ACE inscribe $ABCDE$, half of a regular polygon with an even number of sides. Denote its apothem by a, the area of the surface generated by the semiperimeter as it revolves about AE as an axis by S', and the volume of the solid generated by semipolygon $ABCDE$ by V'.	1. § 517, *a*.
2. Then $V' = S' \cdot \frac{1}{3} a$.	2. § 996, *a*.
3. As the number of sides of the regular polygon, of which $ABCDE$ is half, is repeatedly doubled, V' approaches V as a limit.	3. § 996, *c*.
4. Also S' approaches S as a limit.	4. § 970.
5. And a approaches R as a limit.	5. § 543, I.
6. $\therefore S' \cdot a$ approaches $S \cdot R$ as a limit.	6. § 592.
7. $\therefore S' \cdot \frac{1}{3} a$ approaches $S \cdot \frac{1}{3} R$ as a limit.	7. § 590.
8. But V' is always equal to $S' \cdot \frac{1}{3} a$.	8. Arg. 2.
9. $\therefore V = S \cdot \frac{1}{3} R$. Q.E.D.	9. § 355.

998. Cor. I. *If* V *denotes the volume,* R *the radius, and* D *the diameter of a sphere,*

$$V = \frac{4}{3}\pi R^3 = \frac{1}{6}\pi D^3.$$

999. Cor. II. *The volumes of two spheres are to each other as the cubes of their radii and as the cubes of their diameters.* (Hint. See § 972.)

1000. Historical Note. It is believed that the theorem of § 999 was proved as early as the middle of the fourth century B.C. by Eudoxus, a great Athenian mathematician already spoken of in §§ 809 and 896.

Ex. 1588. Find the volume of a sphere inscribed in a cube whose edge is 8 inches.

Ex. 1589. The volume of a sphere is $1774\frac{2}{3}\pi$ cubic centimeters. Find its surface.

Ex. 1590. Find the radius of a sphere equivalent to a cone with altitude a and radius of base b.

Ex. 1591. Find the radius of a sphere equivalent to a cylinder with the same dimensions as those of the cone in Ex. 1590.

Ex 1592. The metal cone and cylinder in the figure have their altitude and diameter each equal to 2 R, the diameter of the sphere. Place

the sphere in the cylinder, then fill the cone with water and empty it into the cylinder. How nearly is the cylinder filled ? Next fill the cone with water and empty it into the cylinder three times. Is the cylinder filled ?

Ex. 1593. From the results of Ex. 1592 state, in the form of a theorem, the relation of the volume of a sphere: (a) to the volume of a circumscribed cylinder; (b) to the volume of the corresponding cone. Prove these statements.

1001. Historical Note. The problem "To find a sphere equivalent to a given cone or a given cylinder" (Exs. 1590 and 1591), as well as the properties that the volume of a sphere is two thirds of the volume of the circumscribed cylinder and twice the volume of the corresponding cone (Exs. 1592 and 1593), are due to Archimedes. The importance attached to this by the author himself is spoken of more fully in §§ 542 and 974.

Ex. 1594. A bowl whose inner surface is an exact hemisphere is made to hold $\frac{1}{2}$ gallon of water. Find the diameter of the bowl.

Ex. 1595. A sphere 12 inches in diameter weighs 93 pounds. Find the weight of a sphere of the same material 16 inches in diameter.

Ex. 1596. In a certain sphere the area of the surface and the volume have the same numerical value. Find the volume of the sphere.

Ex. 1597. Find the volume of a spherical shell 5 inches thick if the radius of its inner surface is 10 inches.

Ex. 1598. A pine sphere 24 inches in diameter weighs 175 pounds. Find the diameter of a sphere of the same material weighing 50 pounds.

Ex. 1599. The radius of a sphere is R. Find the radius of a sphere whose volume is one half the volume of the given sphere; twice the volume; n times the volume.

1002. Defs. A **spherical sector** is a solid closed figure generated by a sector of a circle revolving about a diameter of the circle as an axis.

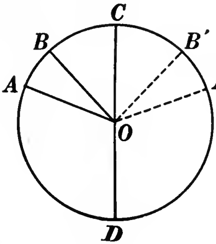


FIG. 1.

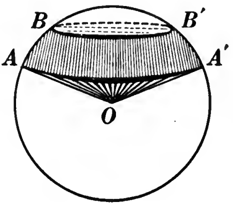


FIG. 2.

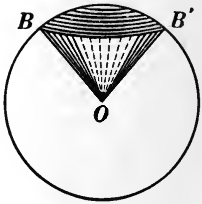


FIG. 3.

The zone generated by the arc of the circular sector is called the **base** of the spherical sector.

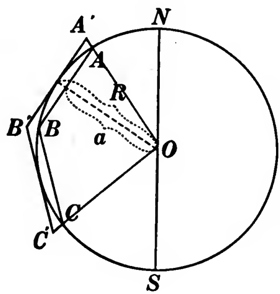
1003. Def. If one radius of the circular sector generating a spherical sector is a part of the axis, *i.e.* if the base of the spherical sector is a zone of one base, the spherical sector is sometimes called a **spherical cone**.

Thus if circular sector AOB (Fig. 1) revolves about diameter CD as an axis, arc AB will generate a zone which will be the base of the spherical sector generated by circular sector AOB (Fig. 2). If circular sector BOC revolves about diameter CD, the spherical sector generated, whose base is the zone generated by arc BC, will be a spherical cone (Fig. 3).

1004. Cor. III. *The volume of a spherical sector is equal to the product of its base and one third the radius of the sphere.*

OUTLINE OF PROOF

Let V denote the volume generated by polygon $OA'B'C'$, v the volume generated by polygon $OABC$, S the area of the surface generated by broken line $A'B'C'$, s the area of the surface generated by broken line ABC, and Z the base of the spherical sector generated by circular sector AOC.



Then $V = S \cdot \frac{1}{3} R$, and $v = s \cdot \frac{1}{3} a$.

$$\therefore \frac{V}{v} = \frac{S \cdot \frac{1}{3} R}{s \cdot \frac{1}{3} a} = \frac{S}{s} \cdot \frac{R}{a}.$$

But $\frac{S}{s} = \frac{R^2}{a^2}$ (Args. 2–5, § 969).

$$\therefore \frac{V}{v} = \frac{R^2}{a^2} \cdot \frac{R}{a} = \frac{R^3}{a^3}.$$

Then by steps similar to § 996, b and c, and § 997, the volume of the spherical sector generated by circular sector $AOC = Z \cdot \frac{1}{3} R$.

1005. Cor. IV. *If V denotes the volume of a spherical sector, Z the area of the zone forming its base, H the altitude of the zone, and R the radius of the sphere,*

$$V = Z \cdot \tfrac{1}{3} R \qquad (\S\ 1004)$$
$$= (H \cdot 2\pi R)\tfrac{1}{3} R \qquad (\S\ 979)$$
$$= \tfrac{2}{3} \pi R^2 H.$$

Ex. 1600. Considering the earth as a sphere with radius R, find the volume of the spherical sector whose base is a zone adjoining the north pole and whose altitude is $\frac{R}{3}$; $\frac{2R}{3}$. Is the one volume twice the other? Compare your results with those of Ex. 1558.

Ex. 1601. Considering the earth as a sphere with radius R, find the volume of the spherical sector whose base is a zone extending: (a) 30° from the north pole; (b) 60° from the north pole. Is the one volume twice the other? Compare your results with those of Ex. 1559.

Ex. 1602. Considering the earth as a sphere with radius R, find the volume of the spherical sector whose base is a zone lying between the parallels of latitude: (*a*) 30° and 45° from the north pole; (*b*) 30° and 45° from the equator. Are the two volumes equal? Compare your results with those of Ex. 1560.

Ex. 1603. Considering the earth as a sphere with radius R, find the area of the zone whose bases are the circumferences of small circles, one 30° north of the equator, the other 30° south of the equator. What part of the entire surface is this zone?

Ex. 1604. What part of the entire volume of the earth is that portion included between the planes of the bases of the zone in Ex. 1603?

HINT. This volume consists of two pyramids and a spherical sector.

Ex. 1605. A spherical shell 2 inches in thickness contains the same amount of material as a sphere whose radius is 6 inches. Find the radius of the outer surface of the shell.

Ex. 1606. A spherical shell 3 inches thick has an outer diameter of 16 inches. Find the volume of the shell.

Ex. 1607. Find the volume of a sphere circumscribed about a rectangular parallelopiped whose edges are 3, 4, and 12.

Ex. 1608. Find the volume of a sphere inscribed in a cube whose volume is 686 cubic centimeters.

Ex. 1609. The surface of a sphere and the surface of a cube are each equal to S. Find the ratio of their volumes. Which is the greater?

Ex. 1610. In a certain sphere the volume and the circumference of a great circle have the same numerical value. Find the surface and the volume of the sphere.

Ex. 1611. How many bullets $\frac{1}{4}$ of an inch in diameter can be made from a sphere of lead 10 inches in diameter? from a cube of lead whose edge is 10 inches?

1006. Defs. A **spherical wedge** is a solid closed figure whose bounding surface consists of a lune and the planes of the sides of the lune.

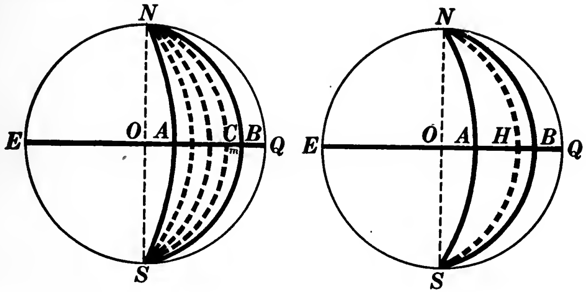
The lune is called the **base** of the spherical wedge, and the angle of the lune the **angle of the spherical wedge.**

1007. Prove, by superposition, the following property of wedges:

In equal spheres, or in the same sphere, two spherical wedges are equal if their angles are equal.

Proposition XXIX. Theorem

1008. *The volume of a spherical wedge is to the volume of the sphere as the number of degrees in the angle of the spherical wedge is to 360.*



Given spherical wedge $NASB$ with the number of degrees in its $\angle$ denoted by N, its volume denoted by W, and the volume of the sphere denoted by V; let $\odot EQ$ be the great $\odot$ whose pole is N.

To prove $\dfrac{W}{V} = \dfrac{N}{360}$.

The proof is left as an exercise for the student.

Hint. The proof is similar to that of § 987.

1009. Cor. I. *The volume of a spherical wedge is equal to the product of its base and one third the radius of the sphere.*

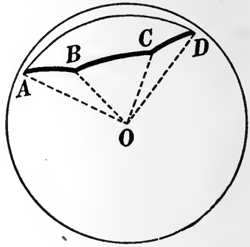
Outline of Proof

$\dfrac{W}{V} = \dfrac{N}{360}$ (§ 1008).

$\therefore\ W = \dfrac{N}{360} \cdot V = \dfrac{N}{360} \cdot S \cdot \frac{1}{3} R = L \cdot \frac{1}{3} R$, where S represents the area of the surface of the sphere, and L the area of the lune, *i.e.* the area of the *base* of the spherical wedge.

Ex. 1612. In a sphere whose radius is 16 inches, find the volume of a spherical wedge whose angle is 40°.

1010. Defs. A **spherical pyramid** is a solid closed figure whose bounding surface consists of a spherical polygon and the planes of the sides of the spherical polygon. The spherical polygon is the **base**, and the center of the sphere the **vertex**, of the spherical pyramid.

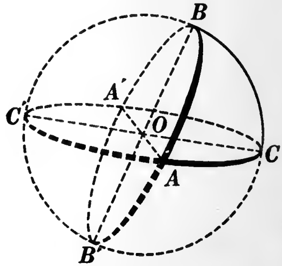


1011. By comparison with § 957, *b* and § 958, prove the following property of spherical pyramids:

In equal spheres, or in the same sphere, two triangular spherical pyramids whose bases are symmetrical spherical triangles are equivalent.

1012. Cor. II. *The volume of a spherical triangular pyramid is equal to the product of its base and one third the radius of the sphere.*

Outline of Proof



1. Pyramid $O\text{–}AB'C' \approx$ pyramid $O\text{–}A'BC$ (§ 1011).

2. $\therefore$ pyramid $O\text{–}ABC$ + pyramid $O\text{–}AB'C' \approx$ wedge $A = 2A \cdot \frac{1}{3}R$ (§ 1009); pyramid $O\text{–}ABC$ + pyramid $O\text{–}AB'C$ = wedge $B = 2B \cdot \frac{1}{3}R$; pyramid $O\text{–}ABC$ + pyramid $O\text{–}ABC'$ = wedge $C = 2C \cdot \frac{1}{3}R$.

3. $\therefore$ twice pyramid $O\text{–}ABC$ + hemispnere $= 2(A + B + C)\frac{1}{3}R$.

4. $\therefore$ twice pyramid $O\text{–}ABC + 360 \cdot \frac{1}{3}R = 2(A + B + C)\frac{1}{3}R$.

5. $\therefore$ pyramid $O\text{–}ABC = (A + B + C - 180)\frac{1}{3}R$

$= \triangle ABC \cdot \frac{1}{3}R = K \cdot \frac{1}{3}R$. Q.E.D.

1013. Cor. III. *The volume of any spherical pyramid is equal to the product of its base and one third the radius of the sphere.* (Hint. Compare with § 805.)

Ex. 1613. Show that the formula of § 997 is a special case of §§ 1004, 1009 and 1013.

Ex. 1614. In a sphere whose radius is 12 inches, find the volume of a spherical pyramid whose base is a triangle with angles 70°, 80°, and 90°.

MISCELLANEOUS EXERCISES ON SOLID GEOMETRY

Ex. 1615. A spherical pyramid whose base is an equiangular pentagon is equivalent to a wedge whose angle is 30°. Find an angle of the base of the pyramid.

Ex. 1616. The volume of a spherical pyramid whose base is an equiangular spherical triangle with angles of 105° is $128\,\pi$ cubic inches. Find the radius of the sphere.

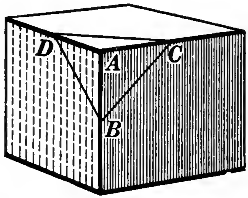
Ex. 1617. In a sphere whose radius is 10 inches, find the angle of a spherical wedge equivalent to a spherical sector whose base has an altitude of 12 inches.

Ex. 1618. Find the depth of a cubical tank that will hold 100 gallons of water.

Ex. 1619. The altitude of a pyramid is H. At what distance from the vertex must a plane be passed parallel to the base so that the part cut off is one half of the whole pyramid? one third? one nth?

Ex. 1620. Allowing 550 pounds of copper to a cubic foot, find the weight of a copper wire $\frac{1}{8}$ of an inch in diameter and 2 miles long.

Ex. 1621. Disregarding quality, and considering oranges as spheres, *i.e.* as similar solids, determine which is the better bargain, oranges averaging $2\frac{3}{4}$ inches in diameter at 15 cents per dozen, or oranges averaging $3\frac{1}{2}$ inches in diameter at 30 cents per dozen.



Ex. 1622. In the figure, B, C, and D are the mid-points of the edges of the cube meeting at A. What part of the whole cube is the pyramid cut off by plane BCD?

HINT. Consider ABC as the base and D as the vertex of the pyramid.

Ex. 1623. Is the result of Ex. 1622 the same if the figure is a rectangular parallelopiped? any parallelopiped?

Ex. 1624. It is proved in calculus that in order that a cylindrical tin can closed at the top and having a given capacity may require the smallest possible amount of tin for its construction, the diameter of the base must equal the height of the can. Find the dimensions of such a can holding 1 quart; 2 gallons.

Ex. 1625. A cylindrical tin can holding 2 gallons has its height equal to the diameter of its base. Another cylindrical tin can with the same capacity has its height equal to twice the diameter of its base. Find the ratio of the amount of tin required for making the two cans. Is your answer consistent with the fact contained in Ex. 1624?

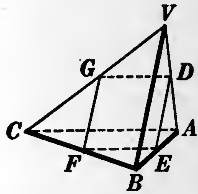
Ex. 1626. A cannon ball 12 inches in diameter is melted, and the lead is cast in the form of a cube. Find the edge of the cube.

Ex. 1627. The cube of Ex. 1626 is melted, and the lead is cast in the form of a cone, the diameter of whose base is 12 inches. Find the altitude of the cone.

Ex. 1628. Find the weight of the cannon ball in Ex. 1626 if a cubic foot of iron weighs 450 pounds.

Ex. 1629. The planes determined by the diagonals of a cube divide the cube into six equal pyramids.

Ex. 1630. Let D, E, F, and G be the mid-points of VA, AB, BC, and CV, respectively, of triangular pyramid $V\text{–}ABC$. Prove $DEFG$ a parallelogram.



Ex. 1631. In the figure, is plane $DEFG$ parallel to edge AC? to edge VB? Prove that any section of a triangular pyramid made by a plane parallel to two opposite edges is a parallelogram.

Ex. 1632. The three lines joining the mid-points of the opposite edges of a tetrahedron bisect each other and hence meet in a point.

Hint. Draw DF and EG. Are these two of the required lines?

Ex. 1633. In a White Mountain two-quart ice cream freezer, the can is $4\frac{5}{8}$ inches in diameter and $6\frac{1}{2}$ inches high; the tub is $6\frac{3}{4}$ inches in diameter at the bottom, 8 inches at the top, and $9\frac{3}{4}$ inches high, inside measurements. (a) Does the can actually hold 2 quarts? (b) How many cubic inches of ice can be packed about the can?

Ex. 1634. Find the total area of a regular tetrahedron whose altitude is a centimeters.

Ex. 1635. The lateral faces of a triangular pyramid are equilateral triangles, and the altitude of the pyramid is 6 inches. Find the total area.

Ex. 1636. In the foundation work of the Woolworth Building, a 55-story building on Broadway, New York City, it was necessary, in order to penetrate the sand and quicksand to bed rock, to sink the caissons that contain the huge shafts of concrete to a depth, in some instances, of 131 feet. If the largest circular caisson, 19 feet in diameter, is 130 feet deep and was filled with concrete to within 30 feet of the surface, how many loads of concrete were required, considering 1 cubic yard to a load?

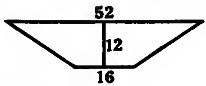
Ex. 1637. From A draw a line meeting line XY in B; let C be the mid-point of AB. Find the locus of C as B moves in line XY.

Ex. 1638. In Ex. 1637, let XY be a plane. Find the locus of C as B moves arbitrarily in plane XY.

Ex. 1639. A granite shaft in the form of a frustum of a square pyramid contains $161\frac{2}{3}$ cubic feet of granite; the edges of the bases are 4 feet and $1\frac{1}{2}$ feet, respectively. Find the height of the shaft.

Ex. 1640. The volume of a regular square pyramid is $42\frac{2}{3}$ cubic feet; its altitude is twice one side of the base. (*a*) Find the total surface of the pyramid; (*b*) find the area of a section made by a plane parallel to the base and one foot from the base.

Ex. 1641. Allowing 1 cubic yard to a load, find the number of loads of earth in a railway cut $\frac{1}{2}$ mile in length, the average dimensions of a cross section being as represented in the figure, the numbers denoting feet. Give the name of the geometrical solid represented by the cut. Why is it not a frustum of a pyramid?

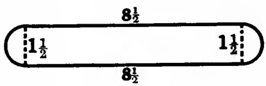


Ex. 1642. For protection against fire, a tank in the form of a frustum of a right circular cone was placed in the tower room of a certain public building. The tank is 16 feet in diameter at the bottom, 12 feet in diameter at the top, and 16 feet deep. If the water in the tank is never allowed to get less than 14 feet deep, how many cubic feet of water would be available in case of an emergency? how many barrels, counting $4\frac{1}{4}$ cubic feet to a barrel?

Ex. 1643. A sphere with radius R is inscribed in a cylinder, and the cylinder is inscribed in a cube. Find: (*a*) the ratio of the volume of the sphere to that of the cylinder; (*b*) the ratio of the cylinder to the cube; (*c*) the ratio of the sphere to the cube.

Ex. 1644. A cone has the same base and altitude as the cylinder in Ex. 1643. Find the ratio of the cone: (*a*) to the sphere; (*b*) to the cylinder; (*c*) to the cube.

Ex. 1645. In a steam-heated house the heat for a room was supplied by a series of 10 radiators each 3 feet high. The average cross section of a radiator is shown in the figure, the numbers denoting inches. It consists of a rectangle with a semicircle at each end. Find the total radiating surface in the room.



Ex. 1646. A coffee pot is 5 inches deep, $4\frac{1}{4}$ inches in diameter at the top, and $5\frac{1}{4}$ inches in diameter at the bottom. How many cups of coffee will it hold, allowing 6 cups to a quart? (Answer to nearest whole number.)

Ex. 1647. Any plane passing through the center of a parallelopiped divides it into two equivalent solids. Are these solids equal?

Ex. 1648. From two points, P and R, on the same side of plane AB, two lines are drawn to point O in plane AB, making equal angles with the plane. Find the locus of point O. (Hint. See Ex. 1237.)

Ex. 1649. A factory chimney is in the form of a frustum of a regular square pyramid. The chimney is 120 feet high, and the edges of its bases are 12 feet and 8 feet, respectively. The flue is 6 feet square throughout. How many cubic feet of material does the chimney contain?

Ex. 1650. Find the edge of the largest cube that can be cut from a regular square pyramid whose altitude is 10 inches and one side of whose base is 8 inches, if one face of the cube lies in the base of the pyramid.

Ex. 1651. Fig. 1 represents a granite monument, the numbers denoting inches. The main part of the stone is 5 feet high, the total height of the stone being 5 feet 6 inches. Fig. 2 represents a view of

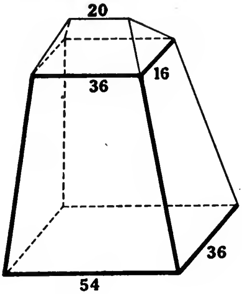


FIG. 1.

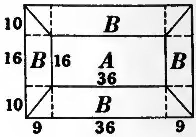


FIG. 2.

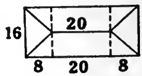


FIG. 3.

the main part of the stone looking directly from above. Fig. 3 represents a view of the top of the stone looking directly from above. Calculate the volume of the stone.

HINT. From Fig. 2 it is seen that the main part of the stone consists of a rectangular parallelopiped A, four right triangular prisms B, and a rectangular pyramid at each corner. Fig. 3 shows that the top consists of a right triangular prism and two rectangular pyramids.

Ex. 1652. The monument in Ex. 1651 was cut from a solid rock in the form of a rectangular parallelopiped. How many cubic feet of granite were wasted in the cutting?

Ex. 1653. In the monument of Ex. 1651 the two ends of the main part, and the top, have a rock finish, the front and rear surfaces of the main part being polished. Find the number of square feet of rock finish and of polished surface.

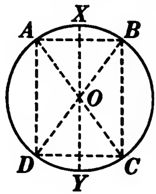
Ex. 1654. The base of a regular pyramid is a triangle inscribed in a circle whose radius is R, and the altitude of the pyramid is $2R$. Find the lateral area of the pyramid.

Ex. 1655. Find the weight in pounds of the water required to fill the tank in Ex. 1323, if a cubic foot of water weighs 1000 ounces.

Ex. 1656. By using the formula obtained in Ex. 1543, find the volume of the sphere inscribed in a regular tetrahedron whose edge is 12.

Ex. 1657. By using the formula obtained in Ex. 1544, find the volume of the sphere circumscribed about a regular tetrahedron whose edge is 12.

Ex. 1658. A hole 6 inches in diameter was bored through a sphere 10 inches in diameter. Find the volume of the part cut out.



HINT. The part cut out consists of two spherical cones and the solid generated by revolving isosceles △ BOC about XY as an axis.

Ex. 1659. Check your result for Ex. 1658 by finding the volume of the part left.

Ex. 1660. Find the area of the spherical surface left in Ex. 1658.

Ex. 1661. Four spheres, each with a radius of 6 inches, are placed on a plane surface in a triangular pile, each one being tangent to each of the others. Find the total height of the triangular pile.

Ex. 1662. Find the total height of a triangular pile of spheres, each with radius of 6 inches, if there are three layers; four layers; n layers.

FORMULAS OF SOLID GEOMETRY

1014. In addition to the notation given in § 761, the following will be used:

$A, B, C, \cdots =$ number of degrees in the angles of a spherical polygon.

$a, b, c, \cdots =$ sides of a spherical polygon.

$B =$ base of spherical sector, wedge, and pyramid.

$C =$ circumference of base in general or of lower base of frustum of cone.

$c =$ circumference of upper base of frustum of cone.

$D =$ diameter of a sphere.

$E =$ spherical excess of a spherical triangle.

$H =$ altitude of zone or spherical sector.

$K =$ area of a spherical triangle or spherical polygon.

$L =$ area of lune.

$N =$ number of degrees in the angle of a lune or wedge.

$R =$ radius of base in general, of lower base of frustum of cone, or of sphere.

$r =$ radius of upper base of frustum of cone.

$S =$ area of surface of a sphere.

$T =$ sum of the angles of a spherical polygon.

$W =$ volume of a wedge.

$Z =$ area of a zone.

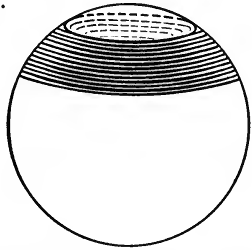
FIGURE	FORMULA	REFERENCE
Prism.	$S = P \cdot E.$	§ 762.
Right prism.	$S = P \cdot H.$	§ 763.
Regular pyramid.	$S = \frac{1}{2} P \cdot L.$	§ 766.
Frustum of regular pyramid.	$S = \frac{1}{2}(P + p)L.$	§ 767.
Rectangular parallelopiped.	$V = a \cdot b \cdot c.$	§ 778.
Cube.	$V = E^3.$	§ 779.
Rectangular parallelopipeds.	$\frac{V}{V'} = \frac{a \cdot b \cdot c}{a' \cdot b' \cdot c'}.$	§ 780.
Rectangular parallelopiped.	$V = B \cdot H.$	§ 782.
Rectangular parallelopipeds.	$\frac{V}{V'} = \frac{B \cdot H}{B' \cdot H'}.$	§ 783.
Any parallelopiped.	$V = B \cdot H.$	§ 790.
Parallelopipeds.	$\frac{V}{V'} = \frac{B \cdot H}{B' \cdot H'}.$	§ 792.
Triangular prism.	$V = B \cdot H.$	§ 797.
Any prism.	$V = B \cdot H.$	§ 799.
Prisms.	$\frac{V}{V'} = \frac{B \cdot H}{B' \cdot H'}.$	§ 801.
Triangular pyramid.	$V = \frac{1}{3} B \cdot H.$	§ 804.
Any pyramid.	$V = \frac{1}{3} B \cdot H.$	§ 805.
Pyramids.	$\frac{V}{V'} = \frac{B \cdot H}{B' \cdot H'}.$	§ 807.
Similar tetrahedrons.	$\frac{V}{V'} = \frac{E^3}{E'^3}.$	§ 812.
Frustum of any pyramid.	$V = \frac{1}{3} H(B + b + \sqrt{B \cdot b}).$	§ 815.
Truncated right triangular prism.	$V = \frac{1}{3} B(E + E' + E'').$	§ 817.
Right circular cylinder.	$S = C \cdot H.$	§ 858.
	$S = 2\pi R \cdot H.$	§ 859.
	$T = 2\pi R(H + R).$	§ 859.
Similar cylinders of revolution.	$\frac{S}{S'} = \frac{H^2}{H'^2} = \frac{R^2}{R'^2}.$	§ 864.
	$\frac{T}{T'} = \frac{H^2}{H'^2} = \frac{R^2}{R'^2}.$	§ 864.
Right circular cone.	$S = \frac{1}{2} C \cdot L.$	§ 873.
	$S = \pi R \cdot L.$	§ 875.
	$T = \pi R(L + R).$	§ 875.
Similar cones of revolution.	$\frac{S}{S'} = \frac{H^2}{H'^2} = \frac{L^2}{L'^2} = \frac{R^2}{R'^2}.$	§ 878.

FIGURE	FORMULA	REFERENCE
Similar cones of revolution.	$\frac{T}{T'} = \frac{H^2}{H'^2} = \frac{L^2}{L'^2} = \frac{R^2}{R'^2}.$	§ 878.
Frustum of right circular cone.	$S = \frac{1}{2}(C + c)\,L.$	§ 882.
	$S = \pi L\,(R + r).$	§ 883.
	$T = \pi L\,(R+r) + \pi\,(R^2 + r^2).$	§ 883.
Cylinder with circular bases.	$V = B \cdot H.$	§ 889.
	$V = \pi R^2 \cdot H.$	§ 890.
Similar cylinders of revolution.	$\frac{V}{V'} = \frac{H^3}{H'^3} = \frac{R^3}{R'^3}.$	§ 891.
Cone with circular base.	$V = \frac{1}{3}\,B \cdot H.$	§ 893.
	$V = \frac{1}{3}\,\pi R^2 \cdot H.$	§ 895.
Similar cones of revolution.	$\frac{V}{V'} = \frac{H^3}{H'^3} = \frac{L^3}{L'^3} = \frac{R^3}{R'^3}.$	§ 897.
Frustum of cone with circular base.	$V = \frac{1}{3}\,H(B + b + \sqrt{B \cdot b}).$	§ 898.
	$V = \frac{1}{3}\,\pi H\,(R^2 + r^2 + R \cdot r).$	§ 899.
Spherical triangle.	$a + b > c.$	§ 941.
Spherical polygon.	$a + b + c + \cdots < 360°.$	§ 942.
Polar triangles.	$A + a' = 180°,\ B + b' = 180°, \cdots$	§ 947.
Spherical triangle.	$A + B + C > 180°$ and $< 540°.$	§ 949.
Sphere.	$S = 4\,\pi R^2.$	§ 971.
Spheres.	$\frac{S}{S'} = \frac{R^2}{R'^2} = \frac{D^2}{D'^2}.$	§ 972.
Zone.	$Z = H \cdot 2\,\pi R.$	§ 979.
Zones.	$\frac{Z}{Z'} = \frac{H}{H'}.$	§ 980.
Lune.	$\frac{L}{S} = \frac{N}{360}.$	§ 987.
	$L = 2\,N.$	§ 988.
Spherical triangle.	$K = (A + B + C) - 180° = E.$	§ 990.
Spherical polygon.	$K = T - (n - 2)\,180.$	§ 993.
Sphere.	$V = S \cdot \frac{1}{3}\,R.$	§ 997.
	$V = \frac{4}{3}\,\pi R^3 = \frac{1}{6}\,\pi D^3.$	§ 998.
Spheres.	$\frac{V}{V'} = \frac{R^3}{R'^3} = \frac{D^3}{D'^3}.$	§ 999.
Spherical sector.	$V = Z \cdot \frac{1}{3}\,R.$	§ 1004.
	$V = \frac{2}{3}\,\pi R^2 \cdot H.$	§ 1005.
Spherical wedge.	$\frac{W}{V} = \frac{N}{360}.$	§ 1008.
	$W = L \cdot \frac{1}{3}\,R.$	§ 1009.
Spherical triangular pyramid.	$V = K \cdot \frac{1}{3}\,R.$	§ 1012.
Any spherical pyramid.	$V = K \cdot \frac{1}{3}\,R.$	§ 1013.

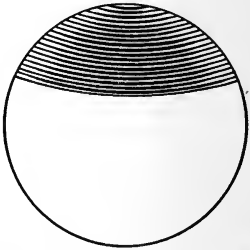
APPENDIX TO SOLID GEOMETRY

SPHERICAL SEGMENTS

1015. Defs. A **spherical segment** is a solid closed figure whose bounding surface consists of a zone and two parallel planes.



Spherical Segment of Two Bases



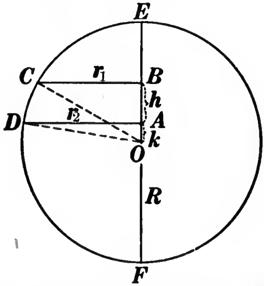
Spherical Segment of One Base

The sections of the sphere formed by the two parallel planes are called the **bases** of the spherical segment.

1016. Defs. State, by aid of §§ 976 and 977, definitions of: (*a*) **Altitude of a spherical segment.** (*b*) **Segment of one base.**

Proposition I. Problem

1017. *To derive a formula for the volume of a spherical segment in terms of the radii of its bases and its altitude.*



Given spherical segment generated by $ABCD$ revolving about EF as an axis, with its volume denoted by V, its altitude by h, and the radii of its bases by r_1 and r_2, respectively.

To derive a formula for V in terms of r_1, r_2, and h.

Draw radii OC and OD. Then $V =$ volume of spherical sector generated by $COD +$ volume of cone generated by $BOC -$ volume of cone generated by AOD.

Denote OA by k, and the radius of the sphere by R.

$$\therefore\ V = \tfrac{2}{3}\pi R^2 h + \tfrac{1}{3}\pi r_1^2(h+k) - \tfrac{1}{3}\pi r_2^2 k$$

$$= \frac{\pi}{3}[(2R^2h + r_1^2 h) + (r_1^2 - r_2^2)k].$$

But $\qquad R^2 = r_2^2 + k^2$; and $R^2 = r_1^2 + (h+k)^2$.

Solving these two equations for R^2 and k,

$$R^2 = \frac{h^4 + r_1^4 + r_2^4 + 2r_1^2h^2 + 2r_2^2h^2 - 2r_1^2r_2^2}{4h^2};\quad k = \frac{r_2^2 - r_1^2 - h^2}{2h}.$$

$$\therefore\ V = \frac{\pi}{3}\,\frac{h^4 + 3r_1^2h^2 + 3r_2^2h^2}{2h} = \frac{\pi h}{2}(r_1^2 + r_2^2) + \tfrac{1}{6}\pi h^3. \qquad \text{Q.E.F.}$$

1018. Cor. I. Problem. *To derive a formula for the volume of a spherical segment of one base:*

(*a*) *In terms of its altitude and the radius of its base;*

(*b*) *In terms of its altitude and the radius of the sphere.*

(*a*) In § 1017, put $r_1 = 0$; then $V = \frac{1}{2}\pi r_2^2 h + \frac{1}{6}\pi h^3$.

(*b*) If h represents the altitude of a segment of one base, and r_2 the radius of the base, then $r_2^2 = h(2R - h)$. § 443, I.

$$\therefore\ V = \tfrac{1}{2}\pi h(2R - h)h + \tfrac{1}{6}\pi h^3 = \pi h^2(R - \tfrac{1}{3}h). \qquad \text{Q.E.F.}$$

Ex. 1663. A dumb-bell consists of the major portion of a sphere with diameter 6 inches attached to each end of a right circular cylinder 12 inches long and 2 inches in diameter. Find the volume of the segment cut from each sphere in fitting it to the cylinder.

Ex. 1664. By means of the formulas given in §§ 1017 and 1018, solve Exs. 1604 and 1658.

THE PRISMATOID

1019. Def. A **prismatoid** is a polyhedron having for bases two polygons in parallel planes, and for lateral faces triangles or trapezoids with one side lying in one base, and the opposite vertex or side lying in the other base, of the polyhedron.

1020. Def. The **altitude** of a prismatoid is the length of the perpendicular from any point in the plane of one base to the plane of the other base.

Proposition II. Problem

1021. *To derive a formula for the volume of a prismatoid.*

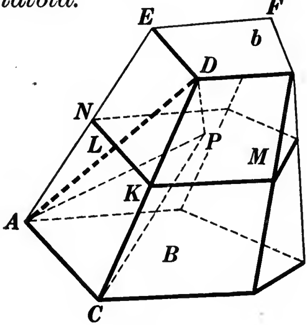


Fig. 1.

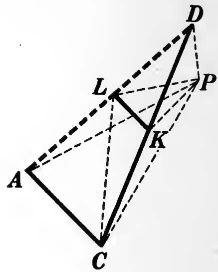


Fig. 2.

Given prismatoid CF with its volume denoted by V, its lower base by B, its upper base by b, its altitude by H, and a section midway between the bases by M.

To derive a formula for V in terms of B, b, H, and M.

If any lateral face as AD is a trapezoid, divide it into two $\triangle$s by diagonal AD, intersecting NK at L.

Let P be any point in M and join it to all vertices of the prismatoid. This will divide the prismatoid into pyramids having their vertices at P and having for their bases B, b, and the triangles forming the lateral faces of the prismatoid.

The volume of pyramid $P\text{–}B = \frac{1}{3} B \cdot \frac{1}{2} H = \frac{1}{6} H \cdot B$; and the volume of pyramid $P\text{–}b = \frac{1}{3} b \cdot \frac{1}{2} H = \frac{1}{6} H \cdot b$. § 805.

Consider pyramid $P\text{–}ADC$. Draw PK, PL, and LC (Fig. 2). This divides pyramid $P\text{–}ADC$ into three pyramids, $D\text{–}KLP$, $C\text{–}KLP$, and $P\text{–}ALC$. Denote $\triangle\, KLP$ by m_1.

Then volume of pyramid $D\text{–}KLP = \frac{1}{6} H \cdot m_1$; and the volume of pyramid $C\text{–}KLP = \frac{1}{6} H \cdot m_1$. § 805.

$$\frac{\text{Pyramid } P\text{–}ALC}{\text{Pyramid } P\text{–}CLK \ (i.e.\ C\text{–}KLP)} = \frac{\triangle\, ALC}{\triangle\, CLK}; \text{ but } \frac{\triangle\, ALC}{\triangle\, CLK} = \frac{AC}{LK} = \frac{2}{1}.$$

$\therefore$ pyramid $P\text{–}ALC \Bumpeq$ twice pyramid $C\text{–}KLP$.

$\therefore$ volume of pyramid $P\text{–}ALC = \frac{2}{6} H \cdot m_1$.

$\therefore$ pyramid $P\text{–}ADC = \frac{1}{6}\, H \cdot m_1 + \frac{1}{6}\, H \cdot m_1 + \frac{2}{6} H \cdot m_1 = \frac{1}{6}\, H \cdot 4\, m_1.$

$\therefore$ the volume of all lateral pyramids $= \frac{1}{6}\, H \cdot 4\, M.$

$\therefore$ $V = \frac{1}{6}\, H \cdot B + \frac{1}{6}\, H \cdot b + \frac{1}{6}\, H \cdot 4\, M = \frac{1}{6}\, H(B + b + 4\, M).$ Q.E.F.

Ex. 1665. By substituting in the prismatoid formula, derive the formula for: (*a*) the volume of a prism (§ 799); (*b*) the volume of a pyramid (§ 805); (*c*) the volume of a frustum of a pyramid (§ 815).

Ex. 1666. Solve Ex. 1651 by applying the prismatoid formula to each part of the monument.

SIMILAR POLYHEDRONS *

1022. The student should prove the following:

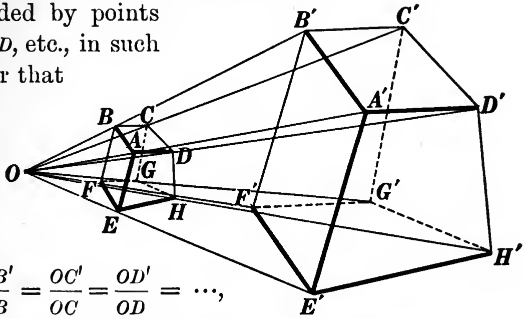
(*a*) *Any two homologous edges of two similar polyhedrons have the same ratio as any other two homologous edges.*

(*b*) *Any two homologous faces of two similar polyhedrons have the same ratio as the squares of any two homologous edges.*

(*c*) *The total surfaces of two similar polyhedrons have the same ratio as the squares of any two homologous edges.*

1023. Def. The **ratio of similitude** of two similar polyhedrons is the ratio of any two homologous edges.

1024. Def. If two polyhedrons $ABCD \cdots$ and $A'B'C'D' \cdots$ are so situated that lines from a point O to A', B', C', D', etc., are divided by points A, B, C, D, etc., in such a manner that



$$\frac{OA'}{OA} = \frac{OB'}{OB} = \frac{OC'}{OC} = \frac{OD'}{OD} = \cdots,$$

the two polyhedrons are said to be **radially placed.**

Ex. 1667. Construct two polyhedrons radially placed and so that point O lies between the two polyhedrons; within the two polyhedrons.

* See § 811. In this discussion only convex polyhedrons will be considered.

PROPOSITION III. THEOREM

1025. *Any two radially placed polyhedrons are similar.* (See Fig. 2 below.)

Given polyhedrons EC and $E'C'$ radially placed with respect to point O.

To prove polyhedron $EC \sim$ polyhedron $E'C'$.

AB, BC, CD, and DA are $\parallel$ respectively to $A'B'$, $B'C'$, $C'D'$, and $D'A'$. § 415.

$\therefore ABCD \parallel A'B'C'D'$, and is similar to it. § 756, II.

Likewise each face of polyhedron EC is $\sim$ to the corresponding face of polyhedron $E'C'$, and the faces are similarly placed.

Again, face $AH \parallel$ face $A'H'$, and face $AF \parallel$ face $A'F'$.

$\therefore$ dihedral $\angle AE =$ dihedral $\angle A'E'$.

Likewise each dihedral $\angle$ of polyhedron EC is equal to its corresponding dihedral $\angle$ of polyhedron $E'C'$.

$\therefore$ each polyhedral $\angle$ of polyhedron EC is equal to its corresponding polyhedral $\angle$ of polyhedron $E'C'$. § 18.

$\therefore$ polyhedron $EC \sim$ polyhedron $E'C'$. § 811. Q.E.D.

PROPOSITION IV. THEOREM

1026. *Any two similar polyhedrons may be radially placed.*

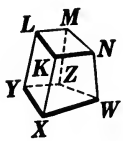


FIG. 1.

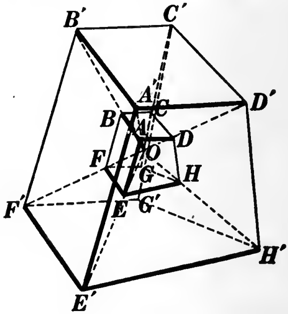


FIG. 2.

Given two similar polyhedrons XM and $E'C'$.

To prove that XM and $E'C'$ may be radially placed.

Outline of Proof

1. Take any point O within polyhedron $E'C'$ and construct polyhedron EC so that it is radially placed with respect to $E'C'$ and so that $OA' : OA = OB' : OB = \cdots = A'B' : KL$.

2. Then polyhedron $EC \sim$ polyhedron $E'C'$. § 1025.

3. Prove that the dihedral $\angle$s of polyhedron EC are equal, respectively, to the dihedral $\angle$s of polyhedron XM, each being equal, respectively, to the dihedral $\angle$s of polyhedron $E'C'$.

4. Prove that the faces of polyhedron EC are equal, respectively, to the faces of polyhedron XM.

5. Prove, by superposition, that polyhedron $EC = XM$.

6. $\therefore$ polyhedron XM may be placed in the position of EC.

7. But EC and $E'C'$ are radially placed.

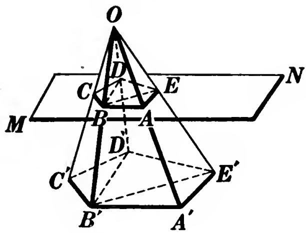
8. $\therefore$ XM and $E'C'$ may be radially placed. Q.E.D.

Proposition V. Theorem

1027. *If a pyramid is cut by a plane parallel to its base:*

I. *The pyramid cut off is similar to the given pyramid.*

II. *The two pyramids are to each other as the cubes of any two homologous edges.*

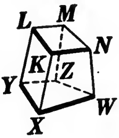


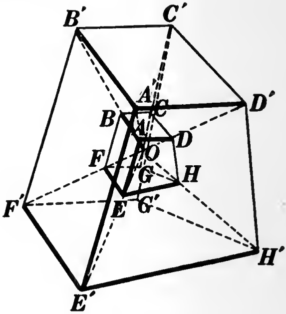
The proofs are left as exercises for the student.

Hint. For the proof of II, pass planes through OB' and diagonals $B'D'$, $B'E'$, etc., dividing each of the pyramids into triangular pyramids. Then pyramid $O\text{–}BCD \sim$ pyramid $O\text{–}B'C'D'$; pyramid $O\text{–}EBD \sim$ pyramid $O\text{–}E'B'D'$, etc. Use § 812 and a method similar to that used in § 505.

PROPOSITION VI. THEOREM.

1028. *Two similar polyhedrons are to each other as the cubes of any two homologous edges.*





Given two similar polyhedrons XM and $E'C'$, with their volumes denoted by V and V', respectively, and with KL and $A'B'$ two homol. edges.

To prove $\dfrac{V}{V'}=\dfrac{\overline{KL}^3}{\overline{A'B'}^3}$.

Place XM in position EC, so that XM and $E'C'$ are radially placed with respect to point O within both polyhedrons. § 1026.

Denote the volumes of pyramids $O\text{–}ABCD$, $O\text{–}AEFB$, etc., by v_1, v_2, etc., and the volumes of pyramids $O\text{–}A'B'C'D'$, $O\text{–}A'E'F'B'$, etc., by v_1', v_2', etc.

Then $\dfrac{v_1}{v_1'}=\dfrac{\overline{AB}^3}{\overline{A'B'}^3}$; $\dfrac{v_2}{v_2'}=\dfrac{\overline{AE}^3}{\overline{A'E'}^3}$; etc. § 1027, II.

But $\dfrac{AB}{A'B'}=\dfrac{AE}{A'E'}=\cdots$ (§ 1022, a); $\therefore \dfrac{\overline{AB}^3}{\overline{A'B'}^3}=\dfrac{\overline{AE}^3}{\overline{A'E'}^3}=\cdots$.

$\therefore \dfrac{v_1}{v_1'}=\dfrac{v_2}{v_2'}=\cdots=\dfrac{\overline{AB}^3}{\overline{A'B'}^3}$; $\therefore \dfrac{v_1+v_2+\cdots}{v_1'+v_2'+\cdots}=\dfrac{\overline{AB}^3}{\overline{A'B'}^3}$. § 401.

$\therefore \dfrac{\text{polyhedron } EC}{\text{polyhedron } E'C'}=\dfrac{\overline{AB}^3}{\overline{A'B'}^3}$; *i.e.* $\dfrac{V}{V'}=\dfrac{\overline{KL}^3}{\overline{A'B'}^3}$. Q.E.D.

1029. Note. Since § 1028 was assumed early in the text (see § 814), the teacher will find plenty of exercises throughout Books VII, VIII, and IX illustrating this principle.

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ROBBINS'S PLANE TRIGONOMETRY

By EDWARD R. ROBBINS, Senior Mathematical Master, William Penn Charter School, Philadelphia, Pa.

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THIS book is intended for beginners. It aims to give a thorough familiarity with the essential truths, and a satisfactory skill in operating with those processes. It is illustrated in the usual manner, but the diagrams are more than usually clear-cut and elucidating.

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